2 The Beam splitter gate has a matrix representation given by

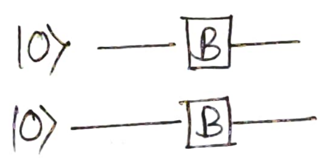
$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

Show that B generates superposition states out of the computational basis states $|0\rangle$ and $|1\rangle$. In particular, show that

$$B \otimes B\,|0\rangle|0\rangle = \left(\frac{i|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{i|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

Show that two applications of the beam splitter gate on the same state, namely that $B(B|\psi\rangle)$ act analogously to the NOT gate, giving the same probabilities of finding $|0\rangle$ and $|1\rangle$.

Solution :- The circuit for this operator will be



$|\psi\rangle = B \otimes B\,|0\rangle|0\rangle$

Since $|\psi\rangle = B \otimes B|0\rangle|0\rangle$

So, $|\psi\rangle \equiv (B|0\rangle)(B|0\rangle)$

Now, $B|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

$\therefore |\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (L.H.S)

Now, the superposition state, $\left(\frac{i|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{i|0\rangle + |1\rangle}{\sqrt{2}}\right)$

$$= \frac{1}{2}\left[\left(i\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\left(i\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right]$$

$$= \frac{1}{2}\left[\left(\begin{pmatrix} j \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} j \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix} j \\ 1 \end{pmatrix} \begin{pmatrix} j \\ 1 \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad (R.H.S)$$

$\therefore$ L.H.S = R.H.S

<u>Showed</u>

4. Quantum gates are universal in the sense that quantum gates can be designed that do anything a classical gate can do. Design a quantum adder, a gate that takes three inputs $|x\rangle$, $|y\rangle$, $|z\rangle$ and that has three output qbits $|x\rangle$, $|x \oplus y\rangle$, $|xy\rangle$, where $|x \oplus y\rangle$ is the sum and $|xy\rangle$ is the carry.

Solution :-

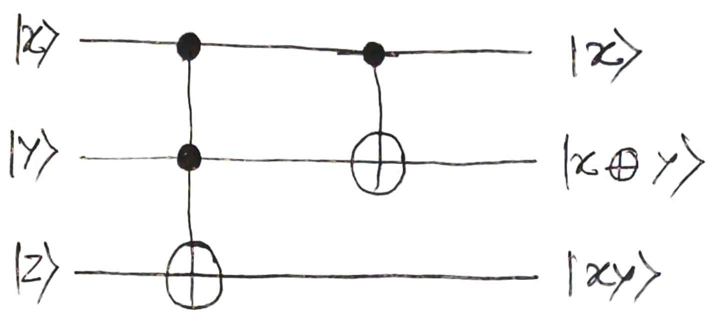
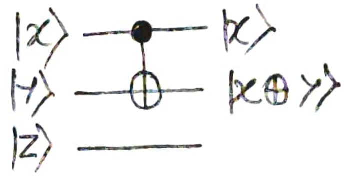


Figure : Quantum adder circuit

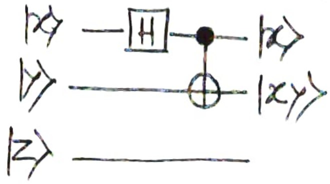
Here, sum is $|x \oplus y\rangle$, so it should not have a carry in Now, $|z\rangle$ should be considered $|0\rangle$ and CNOT gate implem-entation gives $|x \oplus y\rangle$ where $|x\rangle$ is the controlled bit and $|y\rangle$ is the target bit. Now implementing toffoli gate in $|x\rangle$ and $|y\rangle$ we get $|xy\rangle$ where $|x\rangle$ and $|y\rangle$ both are controlled bits and $|0\rangle$ (which means $|z\rangle$) is the target bit. So, it becomes equivalent to AND gate in classical digital circuit, because if $|x\rangle$ and $|y\rangle$ are $|1\rangle$, $|0\rangle$ will be flipped to $|1\rangle$.

Another alternative solution could be,

(a)



(b)



7 Grover's algorithm

Show that the unitary operator corresponding to the phase shift in the Grover iteration is $2|0\rangle\langle 0| - I$.

Solution :

So from Grovers iteration we get that

$$|0\rangle \longrightarrow |0\rangle$$
$$|x\rangle \longrightarrow -|x\rangle \; \forall x > 0$$

Now, from $2|0\rangle\langle 0| - I$, we can show the grovers iteration.

$$\left(2|0\rangle\langle 0| - I\right)|x\rangle$$
$$= 2|0\rangle\langle 0|x\rangle - I|x\rangle$$

Now, lets assume, $x = 0$

$$\text{So, } 2|0\rangle\cancelto{1}{\langle 0|0\rangle} - I|0\rangle$$
$$= 2|0\rangle - |0\rangle$$
$$= |0\rangle$$

Now lets assume $x \neq 0$, so if it is different than $|0\rangle$, it will be orthogonal to $|0\rangle$ then,

$$\text{So, } 2|0\rangle\cancelto{0}{\langle 0|x\rangle} - I|x\rangle$$
$$= -|x\rangle$$

Showed

8. Demonstrate the Hadamard operator on one qbit may be written as

$$H = \frac{1}{\sqrt{2}}\left[\left(|0\rangle + |1\rangle\right)\langle 0| + \left(|0\rangle - |1\rangle\right)\langle 1|\right]$$

Solution:

R.H.S $\frac{1}{\sqrt{2}}\left[\left(|0\rangle + |1\rangle\right)\langle 0| + \left(|0\rangle - |1\rangle\right)\langle 1|\right]$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\0 & -1\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$= H$ L.H.S

$\therefore$ L.H.S = R.H.S

Showed

9. Write out an explicit matrix representation for $H^{\otimes 3}$

Solution :- $H^{\otimes 3}$

$$= H \otimes H \otimes H$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

Ans