Introduction to Quantum Computing

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Contents

I The Quantum Fourier Transform and its application **5**

1 Objectives **9**

2 Activities, materials and more **11**

3 Introduction **13**

3.1 Quantum Circuit that implements QFT 17

3.2 Quantum Phase estimation 18

3.2.1 QPE uses two registers 19

3.3 Summary: QFT and QPE 21

3.3.1 Algorithms 21

Part I

The Quantum Fourier Transform and its application

In this chapter, we talk about ...

1. Modern Physics: Quantum

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Chapter 1

Objectives

The chapter's objective is

General objective

Experiment with simple quantum algorithms in the quantum circuit model.

Chapter 2

Activities, materials and more

In this chapter you will use:

- Computer
- Mobile phone
- tablet

In activities section, we will use:

- Kahoot
- Padlet
- GForms
- etc.

Chapter 3

Introduction

Quantum computers can efficiently perform some tasks which are not feasible on a classical computer.
In this chapter develop the Quantum Fourier Transformation (QFT).

QFT

It is an algorithm

- Quantum
- Efficient

for performing a Fourier transforms of classical data.

Some of the details

- Phase estimation (find unitary operators)
- order-finding problem and factoring problem
- counting solutions (phase estimation and quantum search algorithm)

Options to find solutions,

Transform into known solutions

A great discovery of quantum computation has been that some such transformations can be computed much faster on a quantum computer than on a classical computer, a discovery which has enabled the construction of fast algorithms for quantum computers.

Classical discrete Fourier Transformation

$$y_k \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i j k/N} \tag{3.1}$$

Quantum Fourier Transformation (QFT)

$$|j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{2\pi ijk/N} |k\rangle \tag{3.2}$$

QFT on an orthonormal basis $\{|0\rangle, .., \{|N-1\rangle\}$ is a Linear operator. The action on an arbitrary state

$$\sum_{j=0}^{N-1} x_j |j\rangle \longrightarrow \sum_{k=0}^{N-1} y_k |k\rangle \tag{3.3}$$

where N is the lenght of the vector, x, y are vector of complex numbers (inputs, outputs). We rewrite the eq. (3.2), We will do $N = 2^n$, n us some integer and $\{|0\rangle, .., |2^n - N\rangle\}$ is the computation basis for an n qbit quantum computer.

$$|j\rangle \quad \rightarrow \quad \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi ijk/2^n} |k\rangle \tag{3.4}$$

If we do same for more kets, then if $n = 1$

$$\begin{aligned}
|j\rangle \quad &\rightarrow \quad \frac{1}{2^{1/2}} \sum_{k_1=0}^{1} e^{2\pi ijk_1/2} |k_1\rangle \\
|j\rangle \quad &\rightarrow \quad \frac{1}{2^{1/2}} \sum_{k_2=0}^{1} e^{2\pi ijk_2/2} |k_2\rangle \\
|j\rangle \quad &\rightarrow \quad \frac{1}{2^{1/2}} \sum_{k_3=0}^{1} e^{2\pi ijk_3/2} |k_3\rangle \\
... \quad & ... \quad ... \\
|j\rangle \quad &\rightarrow \quad \frac{1}{2^{m/2}} \sum_{k_m=0}^{1} e^{2\pi ijk_m/2^m} |k_m\rangle
\end{aligned}$$

Note in the argument of the exponential function the summation is over k's. But we want a string of qbits (note we do $m \rightarrow n$)

$$\begin{aligned}
|j_1 j_2 ... j_l\rangle \quad \rightarrow \quad & \frac{1}{2^{1/2}} \sum_{k=0}^{1} e^{2\pi ijk_1/2} \otimes \frac{1}{2^{1/2}} \sum_{k=0}^{1} e^{2\pi ijk_2/2} \otimes \frac{1}{2^{1/2}} \sum_{k=0}^{1} e^{2\pi ijk_3/2} \otimes ... \otimes \\
& \frac{1}{2^{n/2}} \sum_{k=0}^{1} e^{2\pi ij_n k_n/2^n} |k_1 k_2 ... k_n\rangle
\end{aligned} \tag{3.5}$$

Then we can put more qubits as (use eq. (3.4))

$$\begin{aligned} |j_1 j_2 ... j_l\rangle &\rightarrow \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi i j \left(\sum_{l=1}^{n} k_l 2^{-l} \right)} |k_1 \dots k_n\rangle \\ &= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} \bigotimes_{l=1}^{n} e^{2\pi i j k_l 2^{-l}} |k_l\rangle \\ &= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[\sum_{k_l=0}^{1} e^{2\pi i j k_l 2^{-l}} |k_l\rangle \right] \end{aligned} \tag{3.6}$$

Where we used,

$$\begin{aligned} k_n &= \sum_{l=1}^{n} k_l 2^{n-l} \\ k_n 2^{-n} &= \sum_{l=1}^{n} k_l 2^{-l} \end{aligned} \tag{3.7}$$

with $k = k_1 k_2 ... k_n$.
We expand for 0 and 1, and put everything in terms of n

$$\begin{aligned} |j_1 j_2 ... j_n\rangle &\rightarrow \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left[|0\rangle + e^{2\pi i 0 \cdot j 2^{-l}} |1\rangle \right] \\ &= \frac{\left(|0\rangle + e^{2\pi i 0.j_n} |1\rangle \right) \left(|0\rangle + e^{2\pi i 0.j_{n-1} j_n} |1\rangle \right) \cdots \left(|0\rangle + e^{2\pi i 0.j_1 j_2 \cdots j_n} |1\rangle \right)}{2^{n/2}} \end{aligned} \tag{3.8}$$

where (binary fraction) $0 \cdot j_l j_{l+1} ... j_m = j_l/2 + j_{l+1}/4 + \cdots + j_m/2^{m-l+1}$.

1. QFT is unitary
2. Transformation between Computational basis to QFT basis,

$$|\tilde{x}\rangle = QFT\,|x\rangle \tag{3.9}$$

   where $\tilde{x}$ is on Fourier basis and x computational basis (see on the right hand eq. (3.4))

Example 1 $\left\{ |\tilde{0}\rangle, |\tilde{1}\rangle \right\}$ *We take $n = 1$ and the eq.* (3.4)

$$\begin{aligned} |\tilde{0}\rangle &= \frac{1}{2^{1/2}} \sum_{k=0}^{2^1-1} e^{2\pi i \cdot 0 \cdot k/2^1} |k\rangle \\ &= \frac{1}{2^{1/2}} \sum_{k=0}^{1} e^{2\pi i \cdot 0 \cdot k/2^1} |k\rangle \\ &= \frac{1}{2^{1/2}} \sum_{k=0}^{1} |k\rangle \\ &= \frac{1}{2^{1/2}} \left(|0\rangle + |1\rangle \right) \end{aligned}$$

and

$$
\begin{aligned}
\left|\tilde{1}\right\rangle &= \frac{1}{2^{1/2}} \sum_{k=0}^{2^1-1} e^{2\pi i \cdot 1 \cdot k/2^1} \left|k\right\rangle \\
&= \frac{1}{2^{1/2}} \sum_{k=0}^{1} e^{2\pi i k/2} \left|k\right\rangle \\
\left|\tilde{1}\right\rangle &= \frac{1}{2^{1/2}} \Big(\left|0\right\rangle - \left|1\right\rangle \Big)
\end{aligned}
$$

Example 2 *Multiple qbits Consider* $n = 3$

$$
\begin{aligned}
\left|j\right\rangle &= \frac{1}{2^{3/2}} \sum_{k=0}^{2^3-1} e^{2\pi i jk/2^3} \left|k\right\rangle \\
&= \frac{1}{2^{3/2}} \sum_{k=0}^{7} e^{2\pi i jk/2^3} \left|k\right\rangle
\end{aligned} \tag{3.10}
$$

with $n = 3$*, we have*

$$
\begin{array}{cccc}
\left|000\right\rangle & \left|001\right\rangle & \left|010\right\rangle & \left|011\right\rangle \\
\left|100\right\rangle & \left|101\right\rangle & \left|110\right\rangle & \left|111\right\rangle
\end{array} \tag{3.11}
$$

We can use, for example,

$$
\frac{1}{2^3}\Big(\left|100\right\rangle = 2^2(1) + 2^1(0) + 2^0(0) = 4 \Big) \tag{3.12}
$$

$$
\frac{1}{2^3}\Big(\left|101\right\rangle = 2^2(1) + +2^1(0) + 2^0(1) = 5 \Big) \tag{3.13}
$$

Example 3 $n = 3$ *and* $\left|x\right\rangle = \left|5\right\rangle$ *Consider these values for* n *and* $\left|x\right\rangle$*, using eq.* (3.8)

$$
\begin{aligned}
QFT\left|x\right\rangle &= \frac{\Big(\left|0\right\rangle + e^{2\pi i 0.j_1} \left|1\right\rangle \Big)\Big(\left|0\right\rangle + e^{2\pi i 0.j_1 j_2} \left|1\right\rangle \Big)\Big(\left|0\right\rangle + e^{2\pi i 0.j_1 j_2 j_3} \left|1\right\rangle \Big)}{2^{3/2}} \\
&= \frac{\Big(\left|0\right\rangle + e^{2\pi i x 2^{-1}} \left|1\right\rangle \Big)\Big(\left|0\right\rangle + e^{2\pi i x 2^{-2}} \left|1\right\rangle \Big)\Big(\left|0\right\rangle + e^{2\pi i x 2^{-3}} \left|1\right\rangle \Big)}{2^{3/2}} \\
&= \frac{\Big(\left|0\right\rangle + e^{2\pi i 5/2} \left|1\right\rangle \Big)\Big(\left|0\right\rangle + e^{2\pi i 5/2^2} \left|1\right\rangle \Big)\Big(\left|0\right\rangle + e^{2\pi i 5/2^3} \left|1\right\rangle \Big)}{2^{3/2}}
\end{aligned}
$$

where, we used eqs. (3.6), (3.7) *and* (3.13) *(last one is the initial representation state)*

Reminder:

$$
\begin{aligned}
2\pi i \left(\frac{5}{2}\right) &= 2\pi i \left(\frac{2^2}{2} + \frac{1}{2}\right) \\
&= 2\pi i \left(2 + \frac{1}{2}\right)
\end{aligned}
$$

Since, $e^{2\pi i z} = 1$, for any integer z, $e^{2\pi i \left(\frac{5}{2}\right)} = e^{2\pi i \left(2+\frac{1}{2}\right)} = e^{2\pi i \left(\frac{1}{2}\right)} = e^{\pi i} = -1$

Exercise 1 *$n = 3$ and $|x\rangle = |5\rangle$ Use eq. (3.6) to obtain,*

$$QFT\,|x\rangle \;=\; \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \Big[\sum_{k_l=0}^{1} e^{2\pi i x k_l 2^{-l}} \,|k_l\rangle \Big] \tag{3.14}$$

as a tensor product in terms of x.

3.1 Quantum Circuit that implements QFT

Eq. (3.8) is an expression shows each qbit can be given by

$$\begin{array}{lr} |j\rangle \sim & \text{qbit in the QFT basis} \\ |0\rangle + |1\rangle & \text{Hadamard gate} \\ +e^{2\pi 0.j/2^l}\,|1\rangle & \text{Unitary transformation like } R_k. \end{array}$$

Notice, we have two important things,

1. Phase is qbit-dependent
2. Need more components with more 1 (see the second term with the exponential function).
3. We need Hadamard gates.
4. We need phase gate.

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix} \tag{3.15}$$

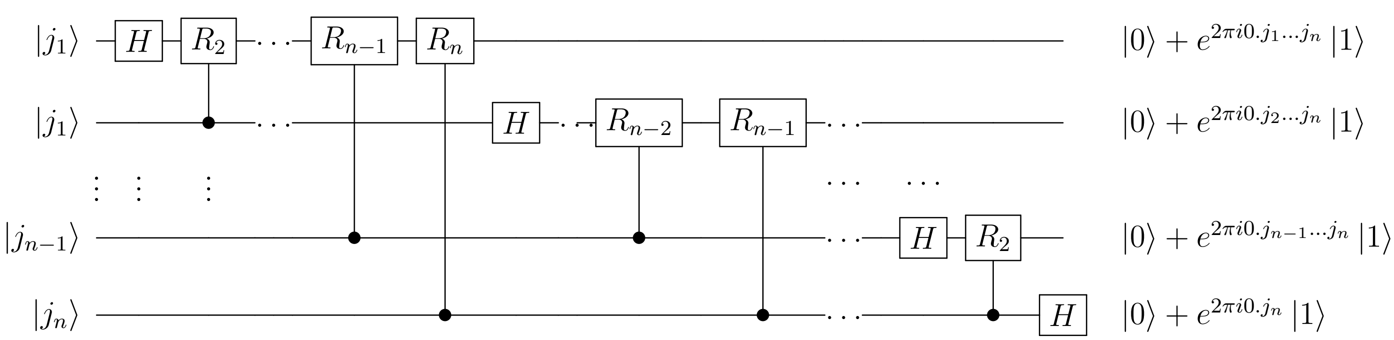

Figure 3.1: Quantum circuit for the QFT

Let us describe the algorithm,

1. Inputs $|\psi\rangle = |j_1 j_2 \ldots j_n\rangle$

2. After H gate $|\psi\rangle = \big(\,|0\rangle + e^{2\pi i j_1/2^1}\,|1\rangle\,\big)\,|j_2 ... j_n\rangle$

3. After R_2 $|\psi\rangle = \big(\,|0\rangle + e^{2\pi i (j_1/2^1 + j_2/2^2)}\,|1\rangle\,\big)\,|j_2 ... j_n\rangle$

4. After R_{n-1} $|\psi\rangle = \big(\,|0\rangle + e^{2\pi i (j_{n-1}/2^{n-1} + ... + j_1/2^1)}\,|1\rangle\,\big)\,|j_2 ... j_n\rangle$

5. ...After R_n $|\psi\rangle = \big(\,|0\rangle + e^{2\pi i (j_n/2^n + ... + j_1/2^1)}\,|1\rangle\,\big)\,|j_2 ... j_n\rangle$

   Apply each the controlled-R_n gates adds an extra bit to the phase of the co-efficient of the first $|1\rangle$. At the end, we have

$$\big(\,|0\rangle + e^{2\pi i j_n/2^n}\,|1\rangle\,\big)\,|j_2 j_3 ... j_n\rangle \tag{3.16}$$

   Recall

$$\begin{aligned} x &= 2^{n-1}x_1 + 2^{n-2}x_2 + ... + 2^0 x_n \\ x/2^n &= 2^1 x_1 + ... + 2^n x_n \end{aligned}$$

6. and after step n

$$\big(\,|0\rangle + e^{2\pi i j_n/2^n}\,|1\rangle\,\big)\,|j_2 j_3 ... j_n\rangle \tag{3.17}$$

Exercise 2 *QFT implementation on the NB Go to this website QFT Qiskit and reproduce the notes in you NB. Share by Github jaorduz.*

Exercise 3 *Nielsen and Chuang. Do the 5.4-5.6 exercises, [1, pag. 221].*

3.2 Quantum Phase estimation

Suppose a unitary operator U has an eigenvector $|u\rangle$ with eigenvalue $e^{2\pi i}$, where the value of φ is unknown. The goal of the phase estimation algorithm is to estimate φ.
Namely,

$$O\,|\psi\rangle = e^{i\theta_\varphi}\,|\psi\rangle \tag{3.18}$$

where $e^{i\theta_\varphi}$ are the eigenvalues of O. Question: Can we extract θ_φ given the ability to prepare $|\psi\rangle$? We can use Quantum Phase Estimation (QPE).
Things we should assume

- We available black boxes (oracles) capable of preparing the state $|u\rangle$ and performing the controlled-U^{2^j} operation, for suitable non-negative integers j.
- The use of black boxes indicates that the phase estimation procedure is not a complete quantum algorithm in its own right. It is a complement (think on this as subroutine, module).

3.2.1 QPE uses two registers

1. First register (counting, t)

   (a) The first register contains t qbits initially in the state $|0\rangle$.
   - t depends on
     - The number of digits for the outut in φ
     - with what probability we wish the phase estimation procedure to be successful

   This dependence is natural.

2. Second register

   (a) It starts in $|u\rangle$, and contains as many qubits as is necessary to store $|u\rangle$.
   - First stage:
     - Apply the circuit fig. 3.2.
   - Second stage
     - Apply the inverse quantum Fourier transform on the first register. Use the QFT ($\mathcal{O}(t^2)$).

3. Final stage of QPE is to read out the state of the first register by doing a measurement in the computational basis Fig. 3.3. This is called a trick:

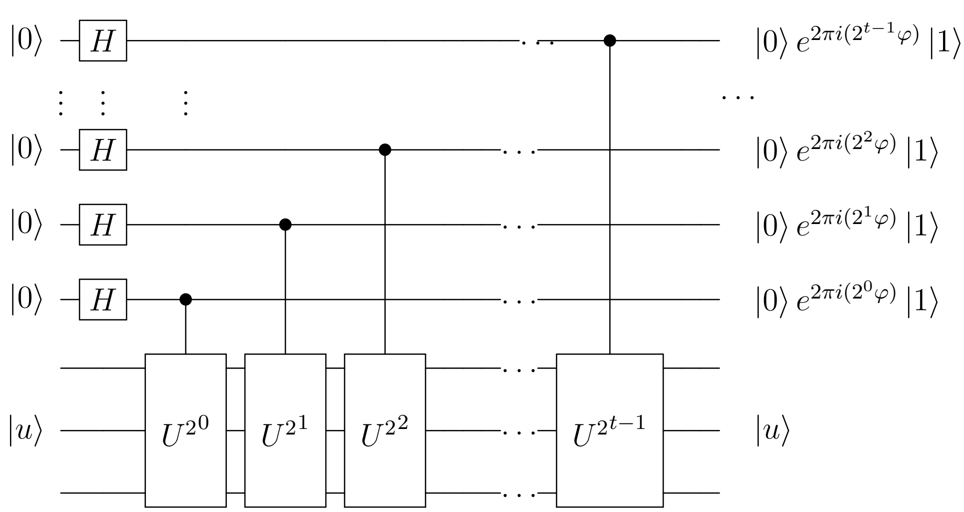

Figure 3.2: Quantum circuit for the QFT

Quantum circuit 3.2 is the representation for

$$\frac{1}{2^{t/2}}\Big(\,|0\rangle + e^{2\pi i 2^{t-1}}\,|1\rangle\Big)\Big(\,|0\rangle + e^{2\pi i 2^{t-2}}\,|1\rangle\Big)...\Big(\,|0\rangle + e^{2\pi i 2^{0}}\,|1\rangle\Big) = \frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1} e^{2i\pi\varphi k}\,|k\rangle \quad (3.19)$$

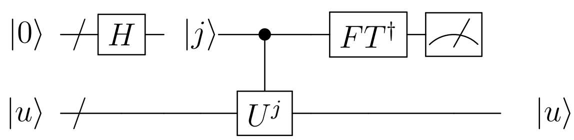

Figure 3.3: Schematics of the overall QPE procedure. / means a bundle of wires

Example 4 *Hc-UH to obtain the phase* *We have the circuit shown in the fig. 3.4.*

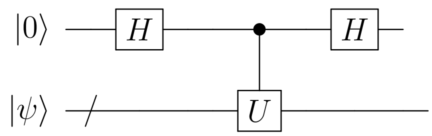

Figure 3.4: An overall schematic of the QPE algorithm (sometimes a.k.a protocol, this is similar to fig. 3.2). H controlled$-U$ and H gates to find the phase. / means a bundle of wires

1. $|0\rangle\,|\psi\rangle$
2. $\frac{1}{\sqrt{2}}\big(\,|0\rangle\,|\psi\rangle + |1\rangle\,|\psi\rangle\,\big)$
3. $\frac{1}{\sqrt{2}}\big(\,|0\rangle\,|\psi\rangle + |1\rangle\,e^{i\theta_\psi}\,|\psi\rangle\,\big)$
4. $\frac{1}{2}\Big(\Big(\,|0\rangle + |1\rangle\,\Big)\,|\psi\rangle + e^{i\theta_\psi}\Big(\,|0\rangle - |1\rangle\,\Big)\,|\psi\rangle\Big) = \frac{1}{2}\Big(\,|0\rangle\,\Big(1+e^{i\theta_\psi}\Big) + |1\rangle\,\Big(1-e^{i\theta_\psi}\Big)\Big)\,|\psi\rangle$

Considering $|\psi\rangle$ is any arbitrary qbit, the previous result (item (4)), we obtain,

Example 5 *Probability* $|0\rangle$

$$\begin{aligned}
P_0 &= \left[\frac{1}{2}\Big(\,\langle 0|\,\Big(1+e^{-i\theta_\psi}\Big) + \langle 1|\,\Big(1-e^{-i\theta_\psi}\Big)\Big)\right]\left[\frac{1}{2}\Big(\,|0\rangle\,\Big(1+e^{i\theta_\psi}\Big) + |1\rangle\,\Big(1-e^{i\theta_\psi}\Big)\Big)\right] \\
&= \left[\frac{1}{2}\Big(\,\langle 0|\,\Big(1+e^{-i\theta_\psi}\Big)\Big)\right]\left[\frac{1}{2}\Big(\,|0\rangle\,\Big(1+e^{i\theta_\psi}\Big)\Big)\right] \\
&= \left|\frac{1}{4}\Big(1+e^{-i\theta_\psi}\Big)\Big(1+e^{i\theta_\psi}\Big)\right| \\
P_0 &= \cos^2\frac{\theta_\psi}{2}
\end{aligned} \tag{3.20}$$

which is the probability to measure $|0\rangle$ *.*

Exercise 4 *Probability* $|1\rangle$ *. Probe* $P_1 = \sin^2\frac{\theta_\psi}{2}$

Previous results, examples 4 and 5, show we need to do a lot of measurements to determine θ_ψ. Therefore, we use the QFT, we get

$$\theta_\psi \to 2\pi\frac{\theta_\psi}{2^n} \tag{3.21}$$

3.3 Summary: QFT and QPE

QFT

1. We went from
   $|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \otimes ... \otimes |x_n\rangle$
   to
   $|\tilde{x}\rangle = \frac{1}{\sqrt{2^n}}(|0\rangle + e^{i2\pi x/2}|1\rangle) \otimes (|0\rangle + e^{i2\pi x/2^2}|1\rangle) \otimes (|0\rangle + e^{i2\pi x/2^3}|1\rangle) \otimes (|\rangle + e^{i2\pi x/2^n}|1\rangle)$

2. We can apply QFT and QFT†

   Quantum Fourier Transform:
   QFT maps an n-qbit input state $|x\rangle$ into an output as

$$\begin{aligned} QFT|x\rangle &= \frac{1}{2^{\frac{n}{2}}}\left(|0\rangle + e^{\frac{2\pi}{2}x}|1\rangle\right) \otimes \left(|0\rangle + e^{\frac{2\pi}{2^2}x}|1\rangle\right) \otimes \\ &\quad \ldots \otimes \left(|0\rangle + e^{\frac{2\pi i}{2^{n-1}}x}|1\rangle\right) \otimes \left(|0\rangle + e^{\frac{2\pi}{2^n}x}|1\rangle\right) \end{aligned} \tag{3.22}$$

QPE

1. It can be though as a module to obtain the phase, considering multiple qbits.

2. This algorithm uses same step 2 in the QFT plus

   Inverse Quantum Fourier Transform (IQFT):
   If we want to recover the state $|2^n\theta\rangle$, we should apply an inverse Fourier transform on the auxiliary register[1]

$$\frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{2\pi i\theta k}|k\rangle \otimes |\psi\rangle \xrightarrow{\mathcal{QFT}_n^{-1}} \frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1} e^{-\frac{2\pi ik}{2^k}(x-2^n\theta)}|x\rangle \otimes |\psi\rangle \tag{3.23}$$

3. after previous steps, we measure

$$|2^n\theta\rangle\,|u\rangle \tag{3.24}$$

   In other words,

$$\frac{1}{2^{n/2}}\sum_{j=0}^{2^n-1} e^{2\pi i\varphi j}\,|j\rangle\,|u\rangle \rightarrow |\tilde{\varphi}\rangle\,|u\rangle\,. \tag{3.25}$$

   where $|\tilde{\varphi}\rangle$ is the estimator for ϕ when measured.

Some requirements are on the LINK

3.3.1 Algorithms

[1] Ref. Qiskit QPE

Algorithm 1 QPE

Require: 1. A black box which performs a controlled-U^j operation, for integer j, 2. an eigenstate $|u\rangle$ of U with eigenvalue $e^{2\pi i\varphi_u}$, and 3. $t = n + \lceil \log 2 + \frac{1}{2\epsilon} \rceil$

Ensure: An $n-$qbit approximation $\tilde{\varphi_u}$ to φ_u

Ensure: $\mathcal{O}(t^2)$ operations and one call to controlled-U^J black box. Succeeds with probability at least $1 - epsilon$.

1: $|0\rangle\,|u\rangle$ ▷ Initial state

2: $\rightarrow \frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} |j\rangle|u\rangle$ ▷ Superposition

3: $\rightarrow \frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} |j\rangle\, U^j\, |u\rangle$ ▷ Oracle

4: $\rightarrow \frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} e^{2\pi i j \varphi_u}\, |j\rangle\, |u\rangle$ ▷ Result oracle

5: $\rightarrow |\tilde{\varphi_u}\rangle\, |u\rangle$ ▷ QFT†

6: $\rightarrow \tilde{\varphi_u}$ ▷

Bibliography

[1] Michael A Nielsen and Isaac Chuang. Quantum computation and quantum information, 2002.