Bloch sphere :

Example 1

Lets say $\theta = \pi$, $\phi = 0$

So, $\vec{r} = \begin{pmatrix} \sin\pi \cdot \cos\phi \\ \sin\pi \cdot \sin\phi \\ \cos\pi \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

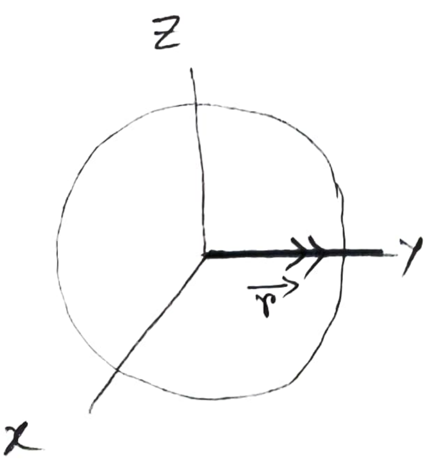
Example 2 Consider the following value : $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$
Obtain $\vec{r}$.

Solution :- $\vec{r} = \begin{pmatrix} \sin\frac{\pi}{2} \cdot \cos\frac{\pi}{2} \\ \sin\frac{\pi}{2} \cdot \sin\frac{\pi}{2} \\ \cos\frac{\pi}{2} \end{pmatrix}$

$= \begin{pmatrix} 1 \cdot 0 \\ 1 \cdot 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Bloch sphere representation



Tensor Product Pdf exercises

Example : Tensor product

Example 7 : Two vectors

$$\begin{pmatrix}1\\2\end{pmatrix} \otimes \begin{pmatrix}4\\8\end{pmatrix} = \begin{pmatrix}4\\8\\8\\16\end{pmatrix}$$

Exercise 2 : $\sigma_x \otimes \sigma_y$

$$= \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \neq \boxed{\begin{pmatrix}0 & -i\\0 & 0\\0 & 0\\i & 0\end{pmatrix}} = \begin{pmatrix}0 & 0 & 0 & -i\\0 & 0 & i & 0\\0 & -i & 0 & 0\\i & 0 & 0 & 0\end{pmatrix}$$

✓

Exercise 3 : $\sigma_y \otimes \sigma_z$

$$= \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$= \boxed{\begin{pmatrix}0 & 0\\0 & i\\i & 0\\0 & 0\end{pmatrix}}$$

Exercise 4 : $\sigma_x \otimes \sigma_z$

$$= \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$$= \boxed{\begin{pmatrix}0 & 0\\0 & -1\\1 & 0\\0 & 0\end{pmatrix}}$$

Exercise 5 Let $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$

write out $|\psi\rangle^{\otimes 4}$

Solution :

$$|\psi\rangle^{\otimes 4} = \left[\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right] \otimes \left[\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right] \otimes \left[\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right] \otimes \left[\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right]$$

$$= \left[\frac{1}{\sqrt{2}} \cdot \left(\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right)\right] \otimes \left[\frac{1}{\sqrt{2}}\left\{\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right\}\right] \otimes \left[\frac{1}{\sqrt{2}}\left\{\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right\}\right] \otimes \left[\frac{1}{\sqrt{2}}\left\{\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right\}\right]$$

$$= \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right] \otimes \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right] \otimes \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right] \otimes \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right]$$

$$= \left[\frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}\right] \otimes \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right] \otimes \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right]$$

$$= \left[\frac{1}{2\sqrt{2}}\begin{pmatrix}1\\-1\\-1\\1\\-1\\1\\1\\-1\end{pmatrix}\right] \otimes \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right]$$

$$= \frac{1}{4}\begin{pmatrix}1\\-1\\-1\\1\\-1\\1\\1\\-1\\-1\\1\\1\\-1\\1\\-1\\-1\\1\end{pmatrix}$$

Exercise 6 Consider next product and figure out if tensor product is commutative.

A) $\sigma_x \otimes \sigma_z = \sigma_z \otimes \sigma_x$

B) $\sigma_x \otimes \sigma_y = \sigma_y \otimes \sigma_z$

C) $(\sigma_x \sigma_y \sigma_z) \otimes \sigma_z = \sigma_x \otimes (\sigma_x \sigma_y \sigma_z)$

Solution:

A) $\sigma_x \otimes \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$

$R.H.S = \sigma_z \otimes \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$

$\therefore$ Yes, it is commutative.

B) $\sigma_x \otimes \sigma_y = \sigma_y \otimes \sigma_x$

$L.H.S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ i & 0 \\ 0 & -i \\ 0 & 0 \end{pmatrix}$

$R.H.S = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -i \\ i & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore$ No, not commutative

c) $(\sigma_x \sigma_y \sigma_z) \otimes \sigma_x = \sigma_x \otimes (\sigma_x \sigma_y \sigma_z)$

$$\therefore \sigma_x \sigma_y \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0+j & 0+0 \\ 0+0 & -j+0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} j+0 & 0+0 \\ 0+0 & 0+j \end{pmatrix}$$

$$= \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$$

$\therefore$ L.H.S =

$$= (\sigma_x \sigma_y \sigma_z) \otimes \sigma_x$$

$$= \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ j & 0 \\ 0 & j \\ 0 & 0 \end{pmatrix}$$

R.H.S

$$= \sigma_x \otimes (\sigma_x \sigma_y \sigma_z)$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & j \\ j & 0 \\ 0 & 0 \end{pmatrix}$$

$\therefore$ No, they are not commutative

Chapter 8

Exercises

1) Represent any complex number in the polar representation (go to Complex plane).

$$r = e^{j\theta} = \cos\theta + i\sin\theta$$

Polar representation

Regular representation/cartesian represantion $r = x + iy$

2) Compute AB and BA considering two matrices (using matrix multiplication)

Ans:

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 4 & 1 \\ 3 & 0 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 6 & 4 & 1 \\ 11 & 10 & 1 \end{pmatrix} \qquad BA = \begin{pmatrix} \; \end{pmatrix}$$

3) Use the definition of dot product to show that 4) Pro
of the following two sets of vectors is orthogon

$$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} \quad \text{and} \quad \left\{\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$$

Ans :

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

Chang state $\langle 0| = \begin{pmatrix}1 & 0\end{pmatrix}$

$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$\{|0\rangle, |1\rangle\}$

Inner product

$\langle 0|1\rangle$

$\langle 1|0\rangle$

I picked $\langle 0|1\rangle$

$= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

$= 0+0$

$= 0$

Lets say,

$|-T\rangle = \begin{pmatrix}-1\\1\end{pmatrix}$

$|+T\rangle = \begin{pmatrix}1\\1\end{pmatrix}$ change of state,

$\langle +T| = \begin{pmatrix}1 & 1\end{pmatrix}$

So, Inner product $\langle +T|-T\rangle$

$= \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}$

$= -1+1$

$= 0$

4) Prove that $U(t_1, t_2) \equiv \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right]$ is Unitary.

Ans :

5. Prove that $\overset{L.H.S}{|\psi(t_2)\rangle} = \overset{R.H.S}{e^{-i\frac{H}{\hbar}(t_2-t_1)}|\psi(t_1)\rangle}$ is solution of the time-independent Shrodinger equation, namely, Excercise

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle$$

Ans:

Shrodinger equation,

$$i\hbar\frac{\partial|\psi_2\rangle}{\partial t_2} = H|\psi_2\rangle$$

So,

$$L.H.S = i\hbar\frac{\partial|\psi_2\rangle}{\partial t_2}$$

$$= i\hbar\frac{\partial}{\partial t_2}\left(e^{-i\frac{H}{\hbar}(t_2-t_1)}|\psi(t_1)\rangle\right)$$

$$= \cancel{i\hbar}\left(-i\frac{H}{\cancel{\hbar}}\right)\left(e^{-i\frac{H}{\hbar}(t_2-t_1)}|\psi(t_1)\rangle\right)$$

$$= H\, e^{-i\frac{H}{\hbar}(t_2-t_1)}|\psi(t_1)\rangle$$

$$= H|\psi(t_2)\rangle \quad (R.H.S)$$

Proved

If he gives us same equation but to prove using t_1,

$$\underbrace{|\psi(t_2)\rangle}_{A} = \overset{e^{-mx}}{e^{-i\frac{H}{\hbar}(t_2-t_1)}}\underbrace{|\psi(t_1)\rangle}_{B}$$

$$\therefore B = e^{mx}A$$

$$\therefore |\psi(t_1)\rangle = e^{i\frac{H}{\hbar}(t_2-t_1)}|\psi(t_2)\rangle$$

$$\text{So, } L.H.S = i\hbar\frac{\partial|\psi_1\rangle}{\partial t_1}$$

$$= i\hbar\frac{\partial}{\partial t_1}\left(e^{i\frac{H}{\hbar}(t_2-t_1)}|\psi(t_2)\rangle\right)$$

$$= -\cancel{i\hbar}\left(-i\frac{H}{\cancel{\hbar}}\right)e^{i\frac{H}{\hbar}(t_2-t_1)}|\psi(t_2)\rangle$$

$$= H e^{i\frac{H}{\hbar}(t_2-t_1)}|\psi(t_2)\rangle$$

$$= H|\psi(t_1)\rangle \quad (R.H.S)$$

$\therefore L.H.S = R.H.S$

Proved

Excercise 6 Prove $|\psi^{00}\rangle \neq |a\rangle \otimes |b\rangle$ for all single state $|a\rangle$ and $|b\rangle$.

Solution:

$$\text{L.H.S} = |\psi^{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left((|0\rangle \otimes |0\rangle) + (|1\rangle \otimes |1\rangle)\right)$$

$$= \frac{1}{\sqrt{2}}\left[\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) + \left(\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right]$$

$|\psi^{00}\rangle = \frac{1}{\sqrt{2}}|00\rangle + |11\rangle$

$\alpha_1|0\rangle + \alpha_2|1\rangle$

$\beta_1|0\rangle + \beta_2|1\rangle$

$$\text{R.H.S} = (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$$

$$= \left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) \otimes \left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right)$$

[assuming $|a\rangle = |0\rangle + |1\rangle$, $|b\rangle = |0\rangle + |1\rangle$]

$$= \begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$\therefore$ L.H.S $\neq$ R.H.S

Proved

Excercise 6

Prove $|\psi^{00}\rangle \neq |a\rangle \otimes |b\rangle$ for all single state $|a\rangle$ and $|b\rangle$.

Solution : Let's say $|a\rangle = |0\rangle$ and $|b\rangle = |1\rangle$

$$\text{L.H.S} = |\psi^{00}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left((|0\rangle \otimes |0\rangle) + (|1\rangle \otimes |1\rangle)\right)$$

$$= \frac{1}{\sqrt{2}}\left[\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) + \left(\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right]$$

$$\text{R.H.S} = |a\rangle \otimes |b\rangle$$

$$= |0\rangle \otimes |1\rangle$$

$$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

[assuming $|a\rangle = |0\rangle$, $|b\rangle = |1\rangle$]

$\therefore$ L.H.S $\neq$ R.H.S

Proved

Exercise :

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

which is measured in $\{|+1\rangle, |-1\rangle\}$ basis.
Now calculate $P(+)$ and $P(-)$.

Solution :

We know that,

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

Change of state,

$$\langle +| = \frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)$$

From born's rule we know,

$$P(x) = |\langle x|\psi\rangle|^2$$

So, $P(+) = |\langle + | \psi\rangle|^2$

$$= \left|\frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right|^2$$

$$= \left|\frac{1}{2}\left(\cancel{\langle 0|0\rangle}^{1} - \cancel{\langle 0|1\rangle}^{0} + \cancel{\langle 1|0\rangle}^{0} - \cancel{\langle 1|1\rangle}^{1}\right)\right|^2$$

$$= \left|\frac{1}{2}(1-1)\right|^2$$

$$= \cancel{1} = 0$$

$P(+) = |\langle +|\psi\rangle|^2$

So, $P(-)$ will be $1 - P(+)$

$$\Rightarrow 1 - 0$$

$$\Rightarrow 1$$

Ans

Class Exercise 2 : It is given that $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$ which is measured in $\{|+\rangle, |-\rangle\}$ basis. Calculate $P(+)$ and $P(-)$.

Solution :- From born's rule we know,

$$P(x) = |\langle x|\psi\rangle|^2$$

$$\therefore P(+) = |\langle +|\psi\rangle|^2$$

$$= \left|\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|) \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right|^2$$

$$= \left|\frac{1}{2}(\overset{1}{\cancel{\langle 0|0\rangle}} + \overset{0}{\cancel{\langle 0|1\rangle}} + \overset{0}{\cancel{\langle 1|0\rangle}} + \overset{1}{\cancel{\langle 1|1\rangle}})\right|^2$$

$$= \left|\frac{1}{\cancel{2}} \cdot \cancel{2}\right|^2$$

$$= 1$$

We know that,

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Change of state,

$$\langle +| = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$$

$$\therefore P(-) = 1 - P(+)$$

$$= 1 - 1$$

$$= 0$$

Ans.

$|x\rangle$ Born's rule :-

$$P(x) = |\langle x|\psi\rangle|^2$$

$$\sum_i P(x_i) = 1$$

Class Example :-

Ex : $|\psi\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + \sqrt{2}\,|1\rangle\right)$ is measured in computational basis.

Calculate $P(0)$, $P(1)$.

Sol^n :- According to born's rule,

[we know, $|0\rangle$
So change of state $\langle 0|$

$$P(x) = |\langle \cancel{x}|\psi\rangle|^2$$

$$P(0) = |\langle 0|\psi\rangle|^2$$

$$= \left|\langle 0|\frac{1}{\sqrt{3}}\left(|0\rangle + \sqrt{2}\,|1\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{3}}\left(\cancel{\langle 0|0\rangle}^{1} + \sqrt{2}\,\cancel{\langle 0|1\rangle}^{0}\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{3}}\right|^2$$

$$= \frac{1}{3}$$

$$\therefore P(1) = 1 - P(0)$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

Anvi

Measurement in QM & Computing :-

$B = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

X-Measurement Hadamard basis

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$|+\rangle$ and $|-\rangle$ are eigenstates of $\sigma_x \rightarrow$ Pauli $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Y-measurement (Left-Right) $= \{|+i\rangle, |-i\rangle\}$

$$|+i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$

$$|-i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

$|+i\rangle$ and $|-i\rangle$ are eigenstates of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Z-measurement (computational basis) $= \{|0\rangle, |1\rangle\}$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$|0\rangle$ and $|1\rangle$ are eigenstates of $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$

$\sigma_x, \sigma_y, \sigma_z$ are basis of group $SU(2)$.
(mathematics)

__Class examples__ Measure $\sigma_x|0\rangle$, $\sigma_x|1\rangle$, $\sigma_y|0\rangle$, $\sigma_y|1\rangle$, $\sigma_z|0\rangle$, $\sigma_z|1\rangle$

Solution :-

- $\sigma_x|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$

- $\sigma_x|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

- $\sigma_y|0\rangle = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ j \end{pmatrix} = j\begin{pmatrix} 0 \\ 1 \end{pmatrix} = j|1\rangle$

- $\sigma_y|1\rangle = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -j \\ 0 \end{pmatrix} = -j\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -j|0\rangle$

- $\sigma_z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

- $\sigma_z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -|1\rangle$

Class

<u>Excercise 5</u> Apply CNOT to $\alpha|10\rangle + \beta|11\rangle$

$$\alpha|10\rangle + \beta|11\rangle$$

$$= \alpha|11\rangle + \beta|10\rangle$$

If there is a 1 first, next bit will be flipped, if 0, next bit stays same

~~De~~

Supporting class examples regarding pauli operators :-

Pauli operations,

Calculate these,

- $$\sigma_x|1\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)\,|1\rangle$$
$$= |0\rangle\underset{1}{\cancel{\langle 1|1\rangle}} + |1\rangle\underset{0}{\cancel{\langle 0|1\rangle}}$$
$$= |0\rangle$$

- $$\sigma_x|0\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)\,|0\rangle$$
$$= |0\rangle\overset{0}{\cancel{\langle 1|0\rangle}} + |1\rangle\langle 0|0\rangle$$
$$= |1\rangle$$

- $$\sigma_y|1\rangle = (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)\,|0\rangle$$
$$= -i\,|0\rangle\overset{0}{\cancel{\langle 1|0\rangle}} + i\,|1\rangle\langle 0|0\rangle$$
$$= i\,|1\rangle$$

- $\sigma_y |0\rangle = \left(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\right)|0\rangle$

$$= -i|0\rangle \underbrace{\cancel{\langle 1|0\rangle}}_{0} + i|1\rangle\langle 0|0\rangle$$

$$= i|1\rangle$$

#

- $\sigma_z |1\rangle = \left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)|1\rangle$

$$= |0\rangle \cancel{\langle 0|1\rangle} - |1\rangle\langle 1|1\rangle$$

$$= -|1\rangle$$

- $\sigma_z |0\rangle = \left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)|0\rangle$

$$= |0\rangle\langle 0|0\rangle - |1\rangle \cancel{\langle 1|0\rangle}$$

$$= |0\rangle$$

Class example

1# $\langle 00|10\rangle =$

2# $\langle 01|10\rangle =$

3# $\langle 01|10\rangle =$

4# $\langle 11|10\rangle =$

5# $|00\rangle\langle 10| =$

6# $|010\rangle\langle 011| =$

7# $\langle 111|001\rangle =$

Class notes

Homework 1 : Prove $\sigma_x \cdot R_y(\theta) \cdot \sigma_x = R_y(-\theta)$

Ans : We know that,

$$R_y(\theta) = e^{-i\frac{\theta}{2}\sigma_y} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}\sigma_y$$

$$= \cos\frac{\theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\frac{\theta}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \cos\frac{\theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \sin\frac{\theta}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix} - \begin{pmatrix} 0 & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$\text{L.H.S} = \sigma_x R_y(\theta) \sigma_x$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \\ \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \quad \text{(L.H.S)}$$

Support Exam

$\underset{\sim\sim\sim}{R.H.S} = R_y(-\theta)$

$= e^{-i\left(\frac{-\theta}{2}\right)\sigma_y}$

$= e^{-\left(-i\frac{\theta}{2}\sigma_y\right)}$

$= \cos\frac{\theta}{2} I + i\sin\frac{\theta}{2}\sigma_y$

$= \cos\frac{\theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin\frac{\theta}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$= \begin{pmatrix} \cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix} + \sin\frac{\theta}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} \cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} 0 & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & 0 \end{pmatrix}$

$= \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$ (R.H.S)

$\therefore$ L.H.S = R.H.S

<u>Proved</u>

Supporting examples of Homework 1 :-

Example 1 :- Prove that $\sigma_x \sigma_y \sigma_x = -\sigma_y$

$$L.H.S = \sigma_x \sigma_y \sigma_x$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0+i & 0+0 \\ 0+0 & -i+0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i+0 & i+0 \\ 0-i & 0+0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$= -1 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= -\sigma_y$$

$\therefore$ L.H.S = R.H.S

Proved

Example

Proving the same thing in another way :-

Solution :-

$$L.H.S = \sigma_x \sigma_y \sigma_x$$

$$= \Big[\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\big(-i\,|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\Big]\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)$$

$$= \Big[|0\rangle\langle 1|\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big) + |1\rangle\langle 0|\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\Big]\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)$$

??

$$= \Big[-i\langle 0|0\rangle\langle 1|1\rangle + i\langle 0|0\rangle\langle 1|1\rangle + (-i)\langle 0|0\rangle\langle 1|1\rangle + i\langle 0|0\rangle\langle 1|1\rangle\Big]\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)$$

$=$

Example 2 : Express H as a product of R_x and R_z rotations and $e^{i\phi}$ for some ϕ.

Solution :

Class

Homework 3 Prove,

$$(i)\ \sigma_j = \sigma_j^{\dagger} = \left(\sigma_j^{T}\right)^{*}$$

$$(ii)\ \sigma_j^{2} = I; \qquad x, y, z = j$$

Solution :-

$$(i)\ \sigma_x = \sigma_x^{\dagger} = \left(\sigma_x^{T}\right)^{*}$$

$$L.H.S = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$R.H.S = \left(\sigma_x^{T}\right)^{*} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{T}\right)^{*} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{*} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\therefore L.H.S = R.H.S$$

$$\sigma_y = \sigma_y^{\dagger} = \left(\sigma_y^{T}\right)^{*}$$

$$L.H.S = \sigma_y = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

$$R.H.S = \left(\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}^{T}\right)^{*} = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}^{*} = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

$$\therefore L.H.S = R.H.S$$

$$\sigma_z = \sigma_z^{\dagger} = \left(\sigma_z^{T}\right)^{*}$$

$$L.H.S = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_z^{\dagger} = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{T}\right)^{*} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{*} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore L.H.S = R.H.S$$

Support Question

(ii) $(\sigma_x)^2 = I$

$$\therefore L.H.S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\therefore L.H.S = R.H.S$

$(\sigma_y)^2 = I$

$$\therefore L.H.S = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 - i^2 & 0 \\ 0 & -i^2+0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$= I$

$\therefore L.H.S = R.H.S$

$(\sigma_z)^2 = I$

$$\therefore L.H.S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\therefore L.H.S = R.H.S$

Supporting example of Homework 3

Question :- Show that $H^{\dagger}H = I$

Solution :- L.H.S $= H^{\dagger} H$

$$= \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right]^{\dagger} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}\cdot\sqrt{2}}\left\{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right\}$$

$$= \frac{1}{2}\begin{pmatrix} 1\cdot1+1\cdot1 & 1\cdot1-1\cdot1 \\ 1\cdot1-1\cdot1 & 1\cdot1+1\cdot1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \frac{1}{\cancel{2}} \times \cancel{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

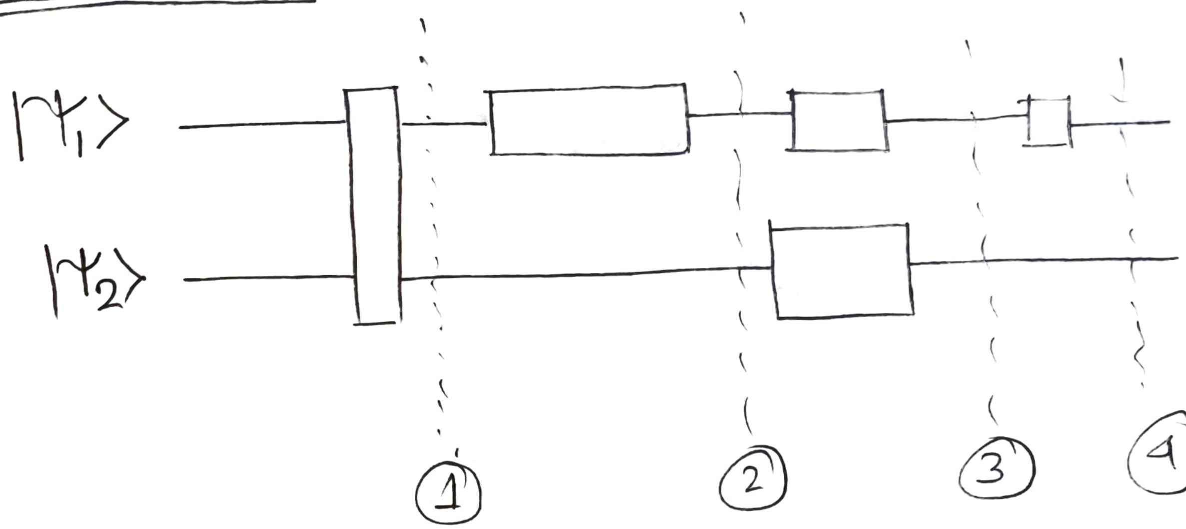
$$= I$$

$$= \text{R.H.S}$$

$\therefore$ L.H.S = R.H.S

Proved

Quantum Gates



Depth = 4

Space = 2

Gates = 5

Answers to questions to discuss =

Question 1 : What is Hilbert space and what is a qbit ?

Ans : Hilbert space = The space that describes the state of a quantum system.

Qbit = The basic unit of quantum information.

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Question 2 : What is an operator ?

Ans : Anything that acts on a qbit is an operator.

Question 3 : What is a Dirac notation ?

Ans : Dirac notation is how to represent a qbit.

Question 4 : What is Shrodinger equation ?

Ans : It is the linear partial differential equation that governs that governs the wave function of a quantum-mechanical system.

$$H|\Psi(t)\rangle = i\hbar \frac{\partial|\Psi(t)\rangle}{\partial t}$$

Question 5 : What does measure mean in QM ?

Ans : In quantum mechanics, the measurement is the testing or manipulation of a physical system in order to yield a numerical result. The predictions that a measurement generates are all probabilistic.

Question 6 : What Kind of probabilities will we use in QM ?

Question : What is tensorial product ?

Ans :

- [illegible]

1. [illegible]
2. Law of [illegible]
3. Law of [illegible]

- Postulates of quantum mechanics :-

Postulate one =

Postulate two =

Postulate three =

Postulate four =