Exercise 4.10 ((x-y) Decomposition of Rotation) Give a decomposition analogous theorem 4.1 but using R_x instead of R_z

Solution :

For a single qbit Unitary operator

$$U = \exp(i\alpha) R_{\hat{n}}(\theta)$$

here,

α = real number

$\hat{n}$ = 3-dimensional unit vector

① Given,

$$U = e^{i\alpha} R_n(\theta) \text{ - - - - } ①$$

To prove that U is unitary operator to satisfy ①

if U is unitary it must satisfy

$$UU^\dagger = I$$

we know,

$$R_{\hat{n}}(\theta) = e^{-i\left(\frac{\theta}{2}\right)\hat{n}.\sigma}$$

Than we can write $U^\dagger = e^{-i\alpha} e^{i\left(\frac{\theta}{2}\right).\hat{n}.\sigma}$

So according to Z-Y Decomposition

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

$$\therefore U \sim R_x(\beta_1) R_y(\gamma_1) R_x(\delta_1)$$

Exercise 4.12 Give A, B, C and α for Hadamard gate.

We know,

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = e^{i\pi/2} R_z(\pi/2) R_y(\pi/2) R_z(\pi/2)$$

From theorem 4.1 we have,

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

Since U is a unitary operator, from (1) and (11) we get,

$$\alpha = \frac{\pi}{2}, \quad \beta = \frac{\pi}{2}, \quad \gamma = \frac{\pi}{2}, \quad \delta = \frac{\pi}{2}$$

We know from corollary theorem,

$$A = R_z(\beta) R_y\left(\frac{\gamma}{2}\right) = R_z\left(\frac{\pi}{2}\right) R_y\left(\frac{\pi}{4}\right)$$

$$B = R_y\left(-\frac{\gamma}{2}\right) R_z\left(-\frac{\delta+\beta}{2}\right)$$

$$C = R_z\left(\frac{\delta-\beta}{2}\right)$$

Exercise 4.13 (circuit identities) It is useful to be able to simplify circuits by inspection, using well-known identities. Prove the following three identities:

$$HXH = Z \; ; \; HYH = -Y \; ; \; HZH = X .$$

- $HXH = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$= \frac{1}{\cancel{2}} \times \cancel{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z = Z$$

- $HYH = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$= \frac{1}{2} \begin{pmatrix} i & -i \\ -i & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix} = \frac{1}{\cancel{2}} \times \cancel{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -Y$$

- $HZH = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \cancel{2} \times \frac{1}{\cancel{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x = X$$

Ab

Exercise 4.14 Use the previous exercise to show that $HTH = R_x(\pi/4)$, up to a global phase.

Solution:

$$HTH \stackrel{?}{=} e^{j\alpha} R_x\left(\frac{\pi}{4}\right)$$

$$\Rightarrow HHTHH \stackrel{?}{=} e^{j\alpha} HR_x(\pi/4)H$$

$$\Rightarrow T \stackrel{?}{=} e^{j\alpha} H\left(\cos\left(\frac{\pi}{8}\right)I - i\sin\left(\frac{\pi}{8}\right)X\right)H$$

$$\Rightarrow T \stackrel{?}{=} e^{j\alpha}\left(\cos\left(\frac{\pi}{8}\right)\cdot \underbrace{HIH}_{I} - i\sin\left(\frac{\pi}{8}\right)\underbrace{HXH}_{Z}\right)$$

$$\Rightarrow T \stackrel{?}{=} e^{j\alpha}\left(\cos\left(\frac{\pi}{8}\right)\cdot I - i\sin\left(\frac{\pi}{8}\right)Z\right)$$

$$\Rightarrow T \stackrel{?}{=} e^{j\alpha}\begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{j\frac{\pi}{8}} \end{pmatrix}$$

Then if we make $\alpha = \frac{\pi}{8}$

$$T = e^{j\frac{\pi}{8}}\begin{pmatrix} e^{-j\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$$

This is true.

Proved

Exercise 4.15 : We can skip it.

...ercise 16 = (Matrix representation of multi-qubit gates)

What is the 4×4 unitary matrix for the circuit in the computational basis?

x_2 —[H]—

x_1 ————

What is the unitary matrix for the circuit (below) in the computational basis

x_2 ————

x_1 —[H]—

Solution :-

(i) x_1 is described by $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

x_2 is described by $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

So, $H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

(ii) x_1 is described by $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

x_2 is described by $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So, $I \otimes H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

* 4.16
4.18
maybe 4.20
(4.21 Don't do that
4.26 **

<u>Solution:-</u>

$|x_1\rangle |x_2\rangle$
$\uparrow \uparrow$ A B

So, left. $X|00\rangle = |00\rangle$
$X|01\rangle = |01\rangle$
$X|10\rangle = |11\rangle$
$X|11\rangle = |10\rangle$

$|x_2\rangle |x_1\rangle$ A B

But for the other one on the right, $X|00\rangle = |00\rangle$
$X|01\rangle = |01\rangle$
$X|10\rangle = |11\rangle$
$X|11\rangle = |10\rangle$

So, here in left when $|x_1\rangle$ is 1 and $|x_2\rangle$ is 0, A & B sets as 11.
But in Right when $|x_1\rangle$ is 1 and $|x_2\rangle$ is 0, A & B sets as 01.
So, they are not the same.

Exercise 16 = (Matrix representation of multi-qbit gates)

What is the 4×4 unitary matrix for the circuit

x_2 —— $\boxed{H}$ ——

x_1 ————————

is the computational basis? What is the Unitary matrix for the circuit

x_2 ————————

x_1 —— $\boxed{H}$ ——

in the computational basis

Solution :-

$|0\rangle$ —— $\boxed{H}$ ——

$|0\rangle$ ————————

$$= H|0\rangle \otimes |0\rangle$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$|0\rangle$ —— $\boxed{H}$ ——

$|1\rangle$ ————————

$$= H|0\rangle \otimes |1\rangle$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

$|1\rangle$ —— $\boxed{H}$ ——

$|0\rangle$ ————————

$$= H|1\rangle \otimes |0\rangle$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$$

$|1\rangle$ —— $\boxed{H}$ ——

$|1\rangle$ ————————

$$= H|1\rangle \otimes |1\rangle$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

$$\begin{array}{c|c}
 & \begin{array}{cccc}|0\rangle\langle 0| & |0\rangle\langle 1| & |1\rangle\langle 0| & |1\rangle\langle 1|\end{array}\\
\begin{array}{c}|0\rangle\langle 0|\\ |0\rangle\langle 1|\\ |1\rangle\langle 0|\\ |1\rangle\langle 1|\end{array} & \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}
\end{array}$$

x_2 ————

x_1 —$\boxed{H}$—

$\|0\rangle$ ———— $\|0\rangle$ —$\boxed{H}$—	$\|0\rangle$ ———— $\|1\rangle$ —$\boxed{H}$—	$\|1\rangle$ ———— $\|0\rangle$ —$\boxed{H}$—	$\|1\rangle$ ———— $\|1\rangle$ —$\boxed{H}$—
$\|0\rangle \otimes H\|0\rangle$ $= \begin{pmatrix}1\\0\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$ $= \begin{pmatrix}1\\0\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\|0\rangle \otimes H\|1\rangle$ $= \begin{pmatrix}1\\0\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$ $= \begin{pmatrix}1\\0\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ $= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\|1\rangle \otimes H\|0\rangle$ $= \begin{pmatrix}0\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$ $= \begin{pmatrix}0\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$	$\|1\rangle \otimes H\|1\rangle$ $= \begin{pmatrix}0\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$ $= \begin{pmatrix}0\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ $= \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$

$$\begin{array}{c|c} & \begin{array}{cccc} |0\rangle\langle 0| & |0\rangle\langle 1| & |1\rangle\langle 0| & |1\rangle\langle 1| \end{array} \\ \begin{array}{c} |0\rangle\langle 0| \\ |0\rangle\langle 1| \\ |1\rangle\langle 0| \\ |1\rangle\langle 1| \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \end{array}$$

A

Example 17 (Building CNOT from controlled-Z gates) Construct a CNOT gate from one controlled-Z gate, that is, the gate whose action in computational basis is specified by the unitary matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and two Hadamard gates, specifying the control and target qbits.

Solution :-

So, we know already, that

$$HZH = X$$

So, we can imply that,

$$CH\ CZ\ CH = CX$$

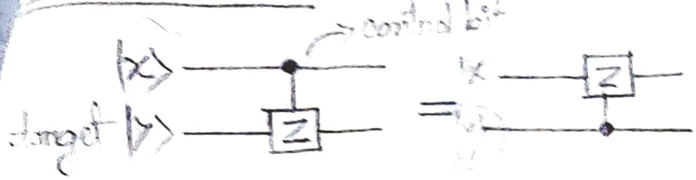
$$\therefore\ CH\ CZ\ CH = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \frac{1}{2}\left[\text{controlled}(2 \times X)\right]$$

$$= \frac{1}{\cancel{2}} \times \cancel{2} \times \text{controlled } X = CNOT$$

ercise 4.18   Show that



L.H.S : Let's say, $x = 0$, $y = 1$

$$
\begin{array}{lll}
 & & CX \\
cz|00\rangle = |00\rangle & = & |00\rangle \\
cz|01\rangle = |01\rangle & = & |01\rangle \\
cz|10\rangle = |10\rangle & = & |11\rangle \\
= cz|11\rangle = -|11\rangle & = & |10\rangle
\end{array}
$$

So, for cz gate, when both the target and control bit is in $|11\rangle$ state, only then it changes to $-|11\rangle$.

R.H.S : Let's say,

$$
\begin{array}{lll}
 & & CX \\
cz|00\rangle = |00\rangle & = & |00\rangle \\
cz|01\rangle = |01\rangle & = & |01\rangle \\
cz|10\rangle = |10\rangle & = & |11\rangle \\
= cz|11\rangle = -|11\rangle & = & |10\rangle
\end{array}
$$

∴ Both the tables are same.

So, L.H.S. = R.H.S

Proved

Exercise 4.19 (CNOT action on density matrix) The CNOT is ~~a~~ simple permutation whose action on a matrix ρ is to rearrange the elements in the matrix. Write out this action explicitly in the computational basis.

Solution:

We know density matrix $\rho = |\psi_{AB}\rangle\langle\psi_{AB}|$

Suppose $|\psi_{AB}\rangle = |00\rangle + |11\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\therefore \langle\psi_{AB}| = \langle 00| + \langle 11| = (1\ 0\ 0\ 1)$

$\therefore$ L.H.S $=$ CNOT(ρ) $= \text{CNOT}\left(|\psi_{AB}\rangle\langle\psi_{AB}|\right)$

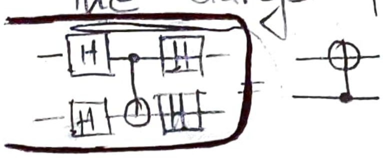
$= \text{CNOT}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}(1\,0\,0\,1)\right\}$

$= \text{CNOT}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is only rearrangment of ρ.

Exercise 20 :- We have described how the CNOT behaves with respect to the computational basis, and in this description the state of the control qbit is not changed. However, if we work in a different basis then the control qbit does change: we will show that its phase is flipped depending on the state of the target qbit! Show that,



Introducing basis states $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. Show this,

$|+\rangle|+\rangle \rightarrow |++\rangle$

$|-\rangle|+\rangle \rightarrow |-+\rangle$

$|+\rangle|-\rangle \rightarrow |--\rangle$

$|-\rangle|-\rangle \rightarrow |+\rangle|-\rangle$

Solution :- $L.H.S = (H \otimes H).\,CX.\,(H \otimes H)$

$$= \left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right\}.\,CX.\left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right\}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.\,CX.\,\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= CNOT_{1,2}$$

In this case, $CNOT_{1,2}|++\rangle = CNOT_{1,2}\left[\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right]$

$$= CNOT_{1,2}\left[\frac{1}{2}\left(\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right)\otimes\left(\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right)\right]$$

$$= CNOT_{1,2}\left[\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}1\\1\end{pmatrix}\right]$$

$$= CNOT_{1,2}\left[\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = |++\rangle$$

In the same way,

$$CNOT_{1,2}|-+\rangle = |-+\rangle$$

$$CNOT_{1,2}|+-\rangle = |--\rangle$$

$$CNOT_{1,2}|--\rangle = |+-\rangle$$

Exercise 4.21 we can skip it

Exercise 4.26

Solution:

Assigning C_1, C_2, T

$R_y(\pi/4) = A$

$R_y(-\pi/4) = B$

$X|C_2 \oplus X|T\rangle$

$|C_2 \oplus T\rangle$

$$|\psi_0\rangle = |C_1\rangle \otimes |C_2\rangle \otimes |T\rangle$$

$$|\psi_1\rangle = |C_1\rangle \otimes |C_2\rangle \otimes A|T\rangle$$

$$|\psi_2\rangle = |C_1\rangle \otimes |C_2\rangle \otimes |(C_2 \oplus A|T\rangle)\rangle \quad //$$

$$|\psi_3\rangle = |C_1\rangle \otimes |C_2\rangle \otimes A\{|(C_2 \otimes A|T\rangle)\rangle\}$$

$$|\psi_4\rangle = |C_1\rangle \otimes |C_2\rangle \otimes |C_1 \oplus A|(C_2 \otimes A|T\rangle)\rangle\rangle$$

$$|\psi_5\rangle = |c_1\rangle \otimes |c_2\rangle \otimes B|c_1 \otimes A|(c_2 \otimes A|T\rangle)\rangle\rangle$$

$$|\psi_6\rangle = |c_1\rangle \otimes |c_2\rangle \otimes |c_2 \oplus B|c_1 \otimes A|(c_2 \otimes A|T\rangle)\rangle\rangle$$

$$|\psi_7\rangle = |c_1\rangle \otimes |c_2\rangle \otimes \underset{=}{B}|c_2 \oplus \underset{=}{B}|c_1 \otimes A|(c_2 \otimes A|T\rangle)\rangle\rangle$$

$$= |c_1\rangle \otimes |c_2\rangle \otimes (-I)|c_2 \oplus (-I)|c_1 \oplus I|c_2 \oplus I|T\rangle\rangle$$

$$= |c_1\rangle \otimes |c_2\rangle \otimes |\cancel{c_2} \oplus |c_1 \oplus |\cancel{c_2} \oplus T\rangle\rangle$$

$$= |c_1\rangle \otimes |c_2\rangle \otimes |c_1 \oplus T\rangle$$

c_1 c_2 T

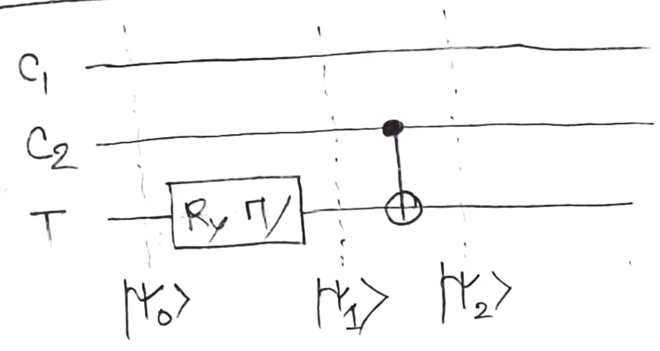
c_1 A	B	C	
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	1 0 0
1	0	1	1 0 1
1	1	0	1 1 1
1	1	1	1 1 0

A	B	C	
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	1 0 1
1	0	1	1 0 0
1	1	0	1 1 1
1	1	1	1 1 0

$|T \oplus c_1 c_2\rangle$

X

4.26



$$|\psi_0\rangle = |C_1\rangle \otimes |C_2\rangle \otimes |T\rangle$$

[Let $R_y(\pi) = A$]

$$|\psi_1\rangle = |C_1\rangle \otimes |C_2\rangle \otimes A|T\rangle$$

$$|\psi_2\rangle = |C_1\rangle \otimes |C_2\rangle \otimes |C_2 \oplus A|T\rangle\rangle$$

$$= |C_1\rangle \otimes |C_2\rangle \otimes \underbrace{|C_2 \oplus |T\rangle\rangle}$$

The result will be dependent on this.

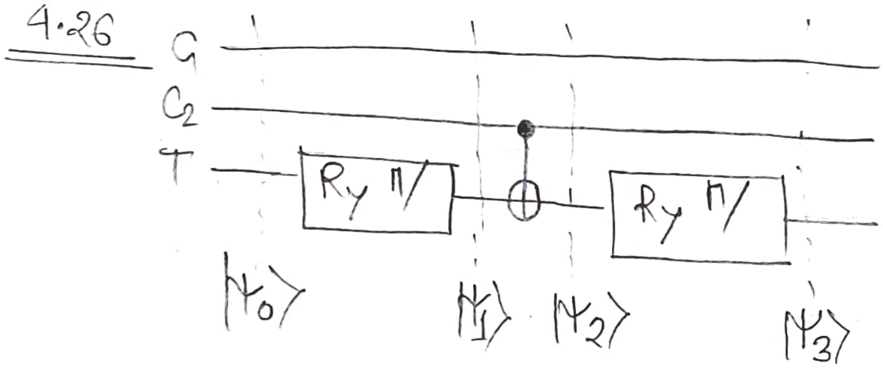
$\|C_1\rangle$	$\|C_2\rangle$	$\|T\rangle$	Toffoli
0	0	0	000
0	0	1	001
0	1	0	010
0	1	1	011
1	0	0	100
1	0	1	101
1	1	0	111
1	1	1	110

$\|C_1\rangle$	$\|C_2\rangle$	$\|T\rangle$	Result
0	0	0	000
0	0	1	001
0	1	0	011
0	1	1	010
1	0	0	100
1	0	1	101
1	1	0	111
1	1	1	110

So, It differs from a toffoli gate only by relative phases. It is same as the toffoli up to relative phases than it is to do the toffoli directly.

(Showed)

4.26



Let $R_y(\pi/) = A$

$$|\psi_0\rangle = |C_1\rangle \otimes |C_2\rangle \otimes |T\rangle$$

$$|\psi_1\rangle = |C_1\rangle \otimes |C_2\rangle \otimes |A|T\rangle\rangle$$

$$|\psi_2\rangle = |C_1\rangle \otimes |C_2\rangle \otimes \big||C_2\rangle \oplus A|T\rangle\big\rangle$$

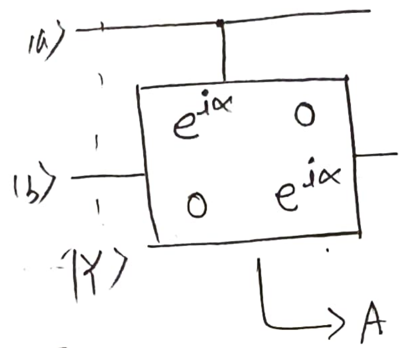
$$|\psi_3\rangle = |C_1\rangle \otimes |C_2\rangle \otimes A\,\big||C_2\rangle \oplus A|T\rangle$$

$$= |C_1\rangle \otimes |C_2\rangle \otimes \underbrace{|C_2\rangle \oplus |T\rangle}_{\text{The result will be dependent on this}}$$

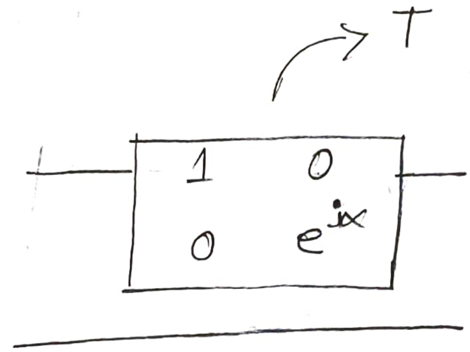
C_1 C_2 T	Toffoli	Result
0 0 0	000	000
0 0 1	001	001
0 1 0	010	011
0 1 1	011	011
1 0 0	100	100
1 0 1	101	101
1 1 0	111	111
1 1 1	110	110

So, it differs from a toffoli gate only by relative phases. It is same as the toffoli up to relative phases than it is to do the toffoli directly.

Prove that

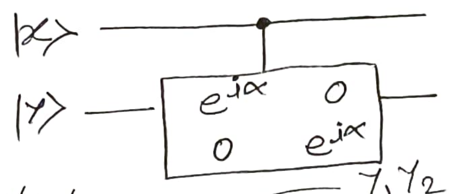


$\stackrel{?}{=}$



Solution :-

So,



(1)

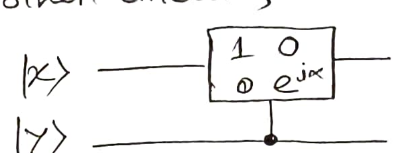
So, $A|\overset{x_1}{0}\overset{x_2}{0}\rangle \longrightarrow |\overset{y_1 y_2}{00}\rangle$

$A|01\rangle \longrightarrow |01\rangle$

$A|10\rangle \longrightarrow e^{i\alpha}|10\rangle$

$A|11\rangle \longrightarrow e^{i\alpha}|11\rangle$

For the other circuit,



(11)

$T|\overset{x_2}{0}\overset{x_1}{0}\rangle \longrightarrow |\overset{y_1 y_2}{00}\rangle$

$T|01\rangle \longrightarrow |01\rangle$

$T|10\rangle \longrightarrow e^{i\alpha}|10\rangle$

$T|11\rangle \longrightarrow e^{i\alpha}|11\rangle$