Introduction to Quantum Computing

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Contents

Introduction

This document contains information about the basics on Quantum Computing.
To check the course information go to this document

Part I

Overview: Classical and Modern Physics

In this part of our course we shall study physics in the realm of atoms, nuclei and elementary particles. These aspects of nature are commonly referred to as quantum phenomena, and we therefore call the subject matter of this volume quantum physics. The currently accepted basic mathematical theory of quantum physics is known as quantum mechanics. In addition, we explore and do some calculations with Dirac's notation.

Chapter 1

Basic concepts on Quantum Mechanics

This chapter shows a particular review about Physics, in particular, we will focus on Quantum Mechanics. We start discussing some facts, experiments and concepts about Classical and Modern Physics. Inside this historical view, we explore concepts as [3, 5]:

1. Classical physics [Go to slides]
2. Newtonian Mechanics [Go to slides]
3. Modern Physics: Quantum

   Part I: [Go to slides]

   Part II: [Go to slides]

   Mathematics and Formalism go to section 4.

Chapter 2

Objectives

The chapter's objective is

General objective

Appraise historical facts and experiments that started the revolutionary change to Quantum Mechanics such as postulates, entanglement, superposition and others, which are used in Quantum Computing.

Chapter 3

Activities, materials and more

In this chapter you will use:

- Computer
- Mobile phone
- tablet

You should enroll in the https://tinyurl.com/yjkytrxs, which is the slack workspace to send any requirement.
In activities section, we will use:

- Kahoot
- Padlet
- GForms
- etc.

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Chapter 4

Mathematics for Quantum Computing

This section exposes the mathematical concepts, some examples and exercises for the students.

4.1 Field

We can think on several experiments with many physical quantities have different values at different points in space.
For instance, the temperature in a room is different at different points: depends on the place you measure the temperature, it will be high or low. In the space, any gravitational force from one massive object, as earth, acting on a body depends on its distance from the earth. The electric field of one point particle goes down as any other particle, with electric charge, goes away. All of those examples are fields, and at every point, there will be some physical quantity to measure.
The term field is used to mean both the region and the value of the physical quantity in the region (for example, electric field, gravitational field). If the physical quantity is a scalar (for example, temperature), we speak of a scalar field. If the quantity is a vector (for example, electric field, force, or velocity), we speak of a vector field [2].
A number system that has addition and multiplication replete with the usual properties is called a field. What we have outlined above is the fact that $\mathbb{C}$ is, like $\mathbb{R}$, a field. When you have a field, you can then create a vector space (see the subsection Vector Space) over that field by taking n-tuples of numbers from that field. Just as we have real n-dimensional vector spaces, $\mathbb{R}^n$, we can as easily create n-dimensional vector spaces over $\mathbb{C}$ which we call $\mathbb{C}^n$.

4.2 Hilbert Space

Considering the definition given in eq. (4.15) in Qbit or Qubit, we allow α_i (with $i = 0, 1$) to be complex. In this vector space (go to Vector Space), we can use different operations (go to Operations).

4.3 Vector Space

Following we discuss the properties, considering a 3D vector [8].
It is customary in mathematics and physics to label an ordered triple of real numbers $\big(x_1, x_2, x_3\big)$ a vector x. The number x_n is called the n th component of vector x. The collection of all such vectors (obeying the properties that follow) form a three-dimensional real vector space. We ascribe five properties to our vectors: If $\mathbf{x} = \big(x_1, x_2, x_3\big)$ and $\mathbf{y} = \big(y_1, y_2, y_3\big)$

1. Vector equality: $\mathbf{x} = \mathbf{y}$ means $x_i = y_i, i = 1, 2, 3$.
2. Vector addition: $\mathbf{x} + \mathbf{y} = \mathbf{z}$ means $x_i + y_i = z_i, i = 1, 2, 3$.
3. Scalar multiplication: $a\mathbf{x} \leftrightarrow \big(ax_1, ax_2, ax_3\big)$ (with a real).
4. Negative of a vector: $-\mathbf{x} = (-1)\mathbf{x} \leftrightarrow \big(-x_1, -x_2, -x_3\big)$.
5. Null vector: There exists a null vector $0 \leftrightarrow (0, 0, 0)$.

As our vector components are real numbers, the following properties also hold:

1. Addition of vectors is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
2. Addition of vectors is associative: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
3. Scalar multiplication is distributive:

$$a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}, \quad \text{also} \quad (a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$$

4. Scalar multiplication is associative: $(ab)\mathbf{x} = a(b\mathbf{x})$.

4.4 Basis

One can find a subset of the vectors which can be used to generate all the other vectors through linear combinations. When we have such a subset that is, in a sense, minimal, we call it a basis for the space.
In $\mathbb{R}^2$, we only need two vectors are needed to produce all the rest, through linear combination. The standard basis is $\mathcal{B}$ for now,

$$\mathcal{B} = \left\{ \{\hat{\mathbf{x}}, \hat{\mathbf{y}}\} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \tag{4.1}$$

With this basis, we can create other vectors.

4.4.1 Properties

A basis is a set of vectors that is linearly independent and complete (spans the space). Basis has two properties:

Linear independence
A set of vectors is linearly independent if we cannot express any one of them as a linear combination of the others. If we can express one as a linear combination of the others, then it is called a linear dependent set. A basis must be linearly independent.

Completeness
It means that the set spans the entire vector space, which in turn means that any vector in the space can be expressed as a linear combination of vectors in that set. A basis must span the space.

Example 1 *An counterexample:*

Let $\mathcal{B}_1$ *a set,*

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \tag{4.2}$$

and the vector

$$\vec{v}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \tag{4.3}$$

Since we are unable to express $\vec{v}_1$ *as a linear combination, then we say* $\mathcal{B}_1$ *is not complete.*

Theorem 1 *The number of elements is the same of the vector space (go to Vector Space) dimension*

Example 2 *Basis In* $\mathbb{C}^2$*, we expand the same* $\mathbf{v} = \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$ *along the basis* $\mathcal{B}$*,*

$$\mathcal{B} \equiv \left\{ \widehat{\mathbf{b}}_0, \widehat{\mathbf{b}}_1 \right\} = \left\{ \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}, \begin{pmatrix} i\sqrt{2}/2 \\ -i\sqrt{2}/2 \end{pmatrix} \right\}$$

First, we confirm that this basis is orthonormal (because the dot-product trick only vorks for orthonormal bases).

$$\begin{aligned} \left\langle \widehat{\mathbf{b}}_0 \middle| \widehat{\mathbf{b}}_1 \right\rangle &= \left(b_{00}\right)^* b_{10} + \left(b_{01}\right)^* b_{11} \\ &= \left(\sqrt{2}/2\right)\left(i\sqrt{2}/2\right) + \left(\sqrt{2}/2\right)\left(-i\sqrt{2}/2\right) \\ &= i/2 - i/2 = 0 \quad \checkmark \end{aligned} \tag{4.4}$$

Also,

$$\left\langle \widehat{\mathbf{b}}_0 \middle| \widehat{\mathbf{b}}_0 \right\rangle = (b_{00})^* b_{00} + (b_{01})^* b_{01} = \left(\sqrt{2}/2\right)\left(\sqrt{2}/2\right) + \left(\sqrt{2}/2\right)\left(\sqrt{2}/2\right) = 1/2 + 1/2 = 1 \quad \checkmark \tag{4.5}$$

and

$$\left\langle \widehat{\mathbf{b}}_1 \middle| \widehat{\mathbf{b}}_1 \right\rangle = (b_{10})^* b_{10} + (b_{11})^* b_{11} = \left(-i\sqrt{2}/2\right)\left(i\sqrt{2}/2\right) + \left(i\sqrt{2}/2\right)\left(-i\sqrt{2}/2\right) = 1/2 + 1/2 = 1 \quad \checkmark \tag{4.6}$$

which establishes orthonormality. We seek

$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}_{\mathcal{B}} \tag{4.7}$$

The "dotting trick" says

$$v_0 = \left\langle \widehat{\mathbf{b}}_0 \middle| \mathbf{v} \right\rangle = (b_{00})^* v_0 + (b_{01})^* v_1 = \left(\sqrt{2}/2\right)\left(1+i\right) + \left(\sqrt{2}/2\right)\left(1-i\right) = \sqrt{2}, \tag{4.8}$$

and

$$v_1 = \left\langle \widehat{\mathbf{b}}_1 \middle| \mathbf{v} \right\rangle = (b_{10})^* v_0 + (b_{11})^* v_1 = \left(-i\sqrt{2}/2\right)\left(1+i\right) + \left(i\sqrt{2}/2\right)\left(1-i\right) = \sqrt{2}, \tag{4.9}$$

so

$$\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}_{\mathcal{B}} \tag{4.10}$$

Finally, we check our work.

$$\sqrt{2}\mathbf{b}_0 + \sqrt{2}\mathbf{b}_1 = \sqrt{2}\begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} + \sqrt{2}\begin{pmatrix} i\sqrt{2}/2 \\ -i\sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} i \\ -i \end{pmatrix} = \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}. \tag{4.11}$$

The dot-product trick for computing expansion coefficients along an orthonormal basis works in this regime, but we have to be careful due to this slight asymmetry. If we want to expand $\mathbf{v}$ *along an orthonormal* $\mathcal{B} = \{\mathbf{b}_k\}$*, we still "dot it" with the individual basis vectors. Say the (as yet unknown) coefficients of* $\mathbf{v}$ *in this basis are* $(\ldots, \beta_k, \ldots)$*. We compute each one using*

$$\langle b_k|v\rangle = \langle b_k| \sum_{j=0}^{n-1} \beta_j |b_j\rangle \quad (4.12)$$

$$= \sum_{j=0}^{n-1} \beta_j \langle b_k|b_j\rangle \quad (4.13)$$

and since orthonormality means

$$\langle b_k|b_j\rangle = \delta_{kj}$$

In the Tensor Algebra, sometimes the δ_{kj} *is called Kronecker delta.*

4.4.2 Complex plane

We already saw that each complex number has two aspects to it: the real term and the term that has the i in it. This creates a natural correspondence between $\mathbb{C}$ and $\mathbb{R}^2$

$$x + iy \quad \leftrightarrow (x, y)$$

As a consequence, a special name is given to Cartesian coordinates when applied to complex numbers: the complex plane. [Advanced Readers. For those of you who already know about vector spaces, real and complex, I'll add a word of caution. This is not the same as a complex vector space consisting of ordered pairs of complex numbers (z, w). The complex plane consists of one point for every complex number, not a point for every ordered pair of complex numbers.]
We'll roll out complex vector spaces in their full glory when we come to the Hilbert space ($\mathcal{H}$) lesson, but it won't hurt to put our cards on the table now. The most useful vector spaces in this course will be ones in which the scalars are the complex numbers. The simplest example is $\mathbb{C}^2$.

Definition 1 *Definition* $\mathbb{C}^2$ *is the set of all ordered pairs of complex numbers,*

$$\mathbb{C}^2 \equiv \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{C} \right\} \quad (4.14)$$

You can verify that this is a vector space and guess its dimension and how the (inner/outer and scalar) product is defined. Then check your guesses online or look ahead in these lectures. All I want to do here is introduce $\mathbb{C}^2$ so you'll be ready to see vectors that have complex components[4, pag. 59].

4.5 Qbit or Qubit

The state of this system is described by a vector in a 2-dimensional Hilbert space. the general state of the system is expressed by the vector

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle \tag{4.15}$$

Equation (4.15) is a state vector in the Hilbert space, which is called ket.

4.6 Dirac's Notation

we define the slates

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which are orthogonal:

$$\langle 0|1\rangle = 1 \cdot 0 + 0 \cdot 1$$

All quantam states are normalized, i.e.

$$\langle \psi|\psi\rangle = 1,$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Qubit is the vector object that will model a physical memory location in a quantum computer. where α_0 and α_1 are complex coefficients, often called the amplitudes of the basis (basis go to Basis) states $|0\rangle$ and $|1\rangle$, respectively.

Definition 2 (Hilbert Space ($\mathcal{H}$)) *It is a complex vector space with following properties [4, pag. 97]:*

- *Inner/outer product (see Scalar product)*
- *is Complete (see Properties. Discussion about completeness in terms of Cauchy sequence is out of scope of this document)*

$\mathcal{H}^2$ is a 2-dimensional vector with complex entries.

Chapter 5

Measurement in Quantum Mechanics and computing

We choose orthogonal basis to describe and measure quantum states. During a measure onto the basis $\{|0\rangle, |1\rangle\}$, the state will collapse into either state $\{|0\rangle$ or $|1\rangle\}$.

X-measurement

As $\{|+\rangle, |-\rangle\}$ are eigenstates of σ_x (which is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, as we will see it). Hadamard (transversal basis states)

$$\Big\{ |+\rangle := \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle \Big), |-\rangle := \frac{1}{\sqrt{2}}\Big(|0\rangle - |1\rangle \Big) \Big\} \tag{5.1}$$

Y-measurement

As $\{|+i\rangle, |-i\rangle\}$ are eigenstates of σ_x (which is $\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, as we will see it). Longitudinal (Left-Right) basis states

$$\Big\{ |+i\rangle := \frac{1}{\sqrt{2}}\Big(|0\rangle + i\,|1\rangle \Big), |-i\rangle := \frac{1}{\sqrt{2}}\Big(|0\rangle - i\,|1\rangle \Big) \Big\} \tag{5.2}$$

Z-measurement

As $\{|0\rangle, |1\rangle\}$ are eigenstates of σ_z (which is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, as we will see it.) Computation basis states,

$$\Big\{ |0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \Big\} \tag{5.3}$$

Born Rule

The probaility that a state $|\psi\rangle$ collapses during a projective measurement onto the basis $\{|x\rangle, |x\rangle^{\perp}\}$ to the state $|x\rangle$ is given by

$$P(x) = |\langle x|\psi\rangle|^{\,2} \tag{5.4}$$

and

$$\sum_i P(x_i) = 1 \tag{5.5}$$

Example 3 (An state in the computational basis) $|\psi\rangle = \frac{1}{\sqrt{3}}\Big(|0\rangle + \sqrt{2}\,|1\rangle\Big)$ *is measured in the basis* $\{|0\rangle, |1\rangle\}$*. Calculate* $P(0)$ *and* $P(1)$*.*

Example 4 (An state in the Hadamard basis) $|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle - |1\rangle\Big)$ *is measured in the basis* $\{|+\rangle, |-\rangle\}$*. Calculate* $P(+)$ *and* $P(-)$*.*

5.1 Bloch Sphere

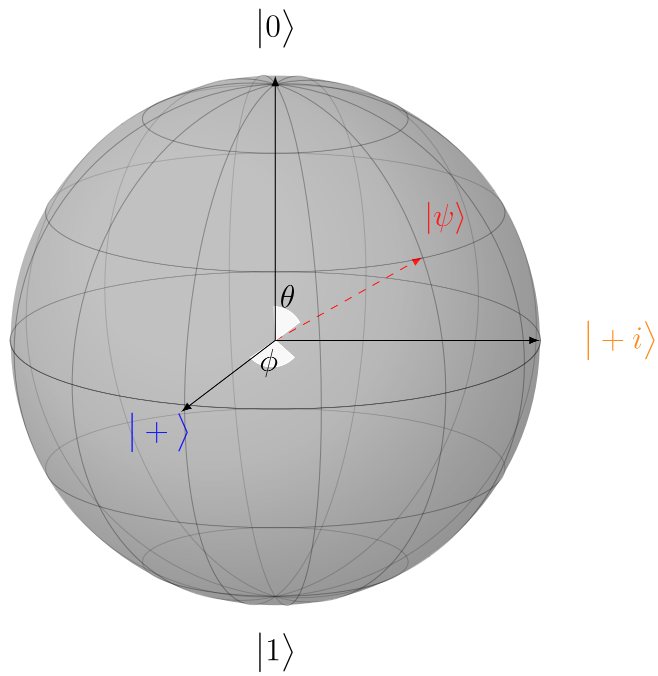


Figure 5.1: Bloch sphere and the representation of $|\psi\rangle$ state vector with their angles and their (computational) basis vectors.

Any normalized (pure) state is given by

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle \tag{5.6}$$

where $0 \leq \phi \leq 2\pi$ is the relative pahse, and $0 \leq \theta \leq \pi$ gives the probability to measure $|0\rangle$ or $|1\rangle$.
Pure states are represented on a sphere surface trough a vector,

$$\vec{r} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \tag{5.7}$$

where $|\vec{r}| = 1$. We call the sphere and vector Bloch (see fig. 5.1).

Example 5 (Bloch) *If we choose* $\theta = 0$*, we obtain*

$$\vec{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.8}$$

which it is considered $|0\rangle$ *. If we choose* $\theta = \pi$*, we obtain*

$$\vec{r} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \tag{5.9}$$

which it is considered $|1\rangle$ *.*

Exercise 1 (Basis on the Bloch sphere) *Consider the following values:*

1. $\theta = \frac{\pi}{2}$ *and* $\phi = 0$*. Obtain* $\vec{r}$*.*
2. $\theta = \frac{\pi}{2}$ *and* $\phi = \pi$*. Obtain* $\vec{r}$*.*
3. $\theta = \frac{\pi}{2}$ *and* $\phi = \frac{\pi}{2}$*. Obtain* $\vec{r}$*.*
4. $\theta = \frac{\pi}{2}$ *and* $\phi = 3\frac{\pi}{2}$*. Obtain* $\vec{r}$*.*

You should use the Bloch sphere to explain your results

Bloch sphere is called a projective sphere because the states of our quantum system are rays in $\mathcal{H}$, and we would prefer to visualize vectors as points, not rays. we go back to the underlying $\mathbb{C}^n$ and project the entire ray (maybe collapse would be a better word) onto the surface of an n -dimensional sphere (whose real dimension is actually $2(n-1)$, but never mind that). We are projecting all those representatives onto a single point on the complex n-sphere. (See Figure 4.5.) Caution: Each point on that sphere still has infinitely many representatives impossible to picture due to a potential scalar factor $e^{i\theta}$, for real θ.]
None of this is to say that scalar multiples, a.k.a. phase changes, never matter. When we start combining vectors in $\mathcal{H}$, their relative phase will become important, and so we shall need to retain individual scalars associated with each component n -tuple. Don't be intimidated; we'll get to that in cautious, deliberate steps [4, pag. 104].

Example 6 *Qubit measurement The measurement of a qubit in the computational basis. This is a measurement on a single qubit with two outcomes defined by the two measurement operators*

$$\begin{aligned} M_0 &= |0\rangle\langle 0| \\ M_1 &= |1\rangle\langle 1| . \end{aligned}$$

Observe that each measurement operator is Hermitian, and that

$$\begin{aligned} M_0^2 &= M_0, \\ M_1^2 &= M_1. \end{aligned}$$

Thus the completeness relation is obeyed,

$$I = M_0^\dagger M_0 + M_1^\dagger M_1 = M_0 + M_1.$$

Suppose the state being measured is

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,.$$

Then the probability of obtaining measurement outcome 0 *is*

$$p(0) = \langle\psi|\, M_0^\dagger M_0\,|\psi\rangle = \langle\psi|\, M_0\,|\psi\rangle = \left|a\right|^2$$

Similarly, the probability of obtaining the measurement outcome 1 *is* $p(1) = |b|^2$*. The state after measurement in the two cases is therefore*

$$\frac{M_0\,|\psi\rangle}{|a|} = \frac{a}{|a|}\,|0\rangle \tag{5.10}$$

$$\frac{M_1\,|\psi\rangle}{|b|} = \frac{b}{|b|}\,|1\rangle\,. \tag{5.11}$$

Chapter 6

Operators

In physics, an observable can be represented by an operator. An operator acts on a ket from the left,

$$A \cdot \big(|\psi_1\rangle \big) = A\,|\psi_1\rangle \tag{6.1}$$

Considering A is not a constant, if not a operator living in Hilbert space, then equation (6.1) is not a simple multiplication.

6.1 Hermitian

An operator, X, is Hermitian adjoint (either Hermitian or Adjoint) if it respects,

$$X = X^\dagger \tag{6.2}$$

Besides the multiplication operations are,

Noncommutative

$$XY \neq YX \tag{6.3}$$

Associative (multiplication)

$$XYZ = X(YZ) = (XY)Z \tag{6.4}$$

Hermitian product

$$\big(XY\big)^\dagger = Y^\dagger X^\dagger \tag{6.5}$$

This is a condition on a matrix to assure the observable being self-adjoint.

Linear (operator) An operator L takes each vector v and transforms it to a new vector Lv. If L is a linear operator, then

$$L(\alpha v + \beta w) = \alpha L v + \beta L w \tag{6.6}$$

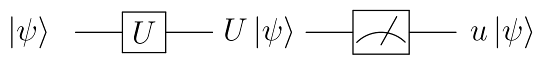
where $\alpha, \beta \in \mathcal{C}$ and $v, w \in \mathcal{H}$.

6.2 Unitary Operator

The reversible operations that a quantum computer can perform upon a single Qbit are represented by the action on the state of the Qbit of any linear transformation that takes unit vectors into unit vectors. Such transformations U are called unitary and satisfy the condition[7, pag. 83][6, pag. 19]:

$$U|\psi\rangle = u|\psi\rangle \tag{6.7}$$

A circuit diagram representing the action on a single Qbit of the 1-Qbit gate U. Initially the Qbit is described by the input state $|\psi\rangle$ on the big. The thin line (wire) represents the subsequent history of the Qbit. After emerging from the box representing u, the Qbit is described on the big by the final state $U|\psi\rangle$.



A unitary operator respects,

$$\boldsymbol{U}\boldsymbol{U}^{\dagger} = \boldsymbol{U}^{\dagger}\boldsymbol{U} = \mathbf{1} \tag{6.8}$$

or

A unitary operator respects,

$$\boldsymbol{U}^{-1} = \boldsymbol{U}^{\dagger} \tag{6.9}$$

Suppose, we have two vector spaces V and W, and we want to know what kind of operator act on the space $V \otimes W$. An operator $A \otimes B$ is linear if,

$$\big(A \otimes B\big)\big(\,|v\rangle \otimes |w\rangle\,\big) \equiv A|v\rangle \otimes B|w\rangle \tag{6.10}$$

6.3 Completeness relation and Unitarity

Readers should go to this document https://tinyurl.com/yzy4ycj5 to read about next properties. Completeness relation is given by,

$$\sum_m M_m^{\dagger} M_m = I \tag{6.11}$$

or, in terms of bracket notation,

$$\sum_i |i\rangle \langle i| = I \tag{6.12}$$

We can use the eq. (6.11) to express next relation

$$1 = \sum_m p(m) = \sum_m \langle \psi| M_m^\dagger M_m |\psi\rangle \tag{6.13}$$

are the completeness and unitarity, respectively. $\{|i\rangle\}$ is any orthonormal basis in $\mathcal{H}$.

6.3.1 Outer product

We use the eq. (6.12) to give a means for representing any operator in the outer product notation.
Suppose $A : V \to W$ is a linear operator, $|v_i\rangle$ is an orthonormal basis for V, and $|w_j\rangle$ an orthonormal basis for W. Using the completness relation twice we obtain

$$\begin{aligned} A &= I_W \; A \; I_V \\ &= \sum_{ij} |w_j\rangle \langle w_j| A |v_i\rangle \langle v_i| \\ &= \sum_{ij} \langle w_j| A |v_i\rangle |w_j\rangle \langle v_i| \end{aligned} \tag{6.14}$$

6.4 Operations

We show some operations in the Quantum computing context.

6.5 State space for two Qbit

Two qbits can be entanglement (two qbits form a single entity). This is an Quantum mechanics concept applied in Classical Computing. Let us give two definitions:

Definition 3 *Definition of Two Qbits A Two-qbit system is (any copy of) the entire product space $\mathcal{H} \otimes \mathcal{H}$.*

Definition 4 *Definition of a Two-Qbit Value The "value" or "state" of a two-qubit system is any unit (or normalized) vector in $\mathcal{H} \otimes \mathcal{H}$.*

where $\mathcal{H}$ is the 2-D Hilbert space of one qbit. We will call two qbits as bipartite system or composite system.

6.5.1 Tensor product

This operation is represented by $\otimes$ operator. Consider two matrices (vectors) A_{mn} and B_{pq}, its tensor product is,

$$A \otimes B \equiv \left.\overbrace{\begin{pmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{pmatrix}}^{nq}\right\}^{mp} \tag{6.15}$$

Then the product $A_{11}B$ denotes A_{11} times B matrix.

Example 7 *Tensor product: Two vectors*

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \times 4 \\ 1 \times 8 \\ 2 \times 4 \\ 2 \times 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 8 \\ 16 \end{pmatrix} \tag{6.16}$$

Consider previous concept to do next exercises.

Exercise 2 *Tensor product with Pauli matrices*

$$\sigma_X \otimes \sigma_Y \tag{6.17}$$

Exercise 3 *Tensor product with Pauli matrices*

$$\sigma_Y \otimes \sigma_Z \tag{6.18}$$

Exercise 4 *Tensor product with Pauli matrices*

$$\sigma_X \otimes \sigma_Z \tag{6.19}$$

We will usually use

$$|\psi\rangle^{\otimes k} \tag{6.20}$$

Eq. (6.20) means $|\psi\rangle$ tensored (tensor operator) k times.

Example 8 $|\psi\rangle -$ *Tensored*

$$|\psi\rangle^{\otimes 3} = |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \tag{6.21}$$

Exercise 5 *Let*

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle \big) \tag{6.22}$$

Write out

- $|\psi\rangle^{\otimes 3}$
- $|\psi\rangle^{\otimes 4}$

in terms of tensor products in the computational basis.

Exercise 6 *Consider next products and figure out if the tensor product is commutative*

- $\sigma_X \otimes \sigma_Z \stackrel{?}{=} \sigma_Z \otimes \sigma_X$
- $\sigma_X \otimes \sigma_Y \stackrel{?}{=} \sigma_X \otimes \sigma_Y$
- $(\sigma_X \sigma_Y \sigma_Z) \otimes \sigma_X \stackrel{?}{=} \sigma_X \otimes (\sigma_X \sigma_Y \sigma_Z)$

Exercise 7 *Probe*

- $$z(|v\rangle \otimes |w\rangle) = (z\,|v\rangle) \otimes |w\rangle = |v\rangle \otimes (z\,|w\rangle).$$
- $$\big(\,|v_1\rangle + |v_2\rangle\,\big) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle\,.$$
- $$|v\rangle \otimes \big(\,|w_1\rangle + |w_2\rangle\,\big) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle\,.$$

Where z is an arbitrary scalar and $|v\rangle\,, |v_1\rangle\,, |v_2\rangle \in V$ and $|w\rangle\,, |w_1\rangle\,, |w_2\rangle \in W$,

6.5.2 Scalar product

The product is defined as you would expect. If

$$|a\rangle = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \quad \text{and} \quad |b\rangle = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}, \tag{6.23}$$

then

$$\langle a|b\rangle \equiv \sum_{k=1}^{n} a_k b_k \tag{6.24}$$

We define,

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{if } k \neq j \end{cases} \tag{6.25}$$

Expressing any vector, $\mathbf{v}$, in terms of a basis.
We call δ_{kj}, the Kronecker delta, is the mathematical way to express anything that is to be 0 unless the index $k = j$, in which case it is 1.
This product is defined between elements of $\mathcal{H}$, which is a linear space. This product of $\phi, \psi \in \mathcal{H}$ is written as $\phi\psi$ (or $\langle\phi|\psi\rangle$). It is just a complex number, not an element of $\mathcal{H}$. This operation has the following properties:

1. Distributive law: $(\phi + \psi)\chi = \phi\chi + \psi$.
2. Associative law: $(\alpha\phi)\psi = \alpha(\phi\psi)$.
3. Hermitian symmetry: $\phi\psi = (\psi\phi)^*$, where * is the complex conjugate.
4. Definite form: $\psi\psi \geq 0$, and $\psi\psi = 0$ only if $\psi = 0$.

This product allows one to define length and distance. The length of the vector ψ is defined as $\sqrt{\psi}\psi$ (it follows from Hermitian symmetry that $\psi\psi$ is real). The distance between ϕ and ψ is the length of the difference of the two vectors[1].

6.5.3 Matrix multiplication

Matrix multiplication is not a commutative operation,

$$AB \neq BA \tag{6.26}$$

It means there are something particular, A_{np} and B_{qm} is possible if it is respectfull $p = q$.

$$A_{np}B_{qm} = C_{nm} \tag{6.27}$$

Example 9 (Two vectors)

$$\begin{pmatrix} 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = 2 + 12 + 30 = 44 \tag{6.28}$$

6.5.4 Linear Transformation

Multiplying a vector by a matrix produces another vector. It is a special kind of mapping that sends vectors to vectors.

6.5.5 Gates

The Pauli gates X, Y, and Z (sometimes they are labeled $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$) correspond to rotations about the $x-, y-$ and $z-$axes of the Bloch sphere (go to subsection 5.1), respectively.

Exercise 8 *Operations*

- *Verify that the Hadamard gate H is unitary $\left(H^\dagger = H^{-1}\right)$*
- *Verify $H^2 = I$*
- *What are the eigenvalues and eigenvectors of H?*

Hint:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{6.29}$$

and

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \tag{6.30}$$

6.6 Computing Eigenvectors and Eigenvalues

An eigenvector of a linear operator A on a vector space is a non-zero vector $|v\rangle$ such that $A|v\rangle = v|v\rangle$, where v is a complex number known as the eigenvalue of A corresponding to $|v\rangle$. It will often be convenient to use the notation v both as a label for the eigenvector, and to represent the eigenvalue. We assume that you are familiar with the elementary properties of eigenvalues and eigenvectors –in particular, how to find them, via the characteristic equation [7, pag. 69]. The characteristic function is defined to be $c(\lambda) \equiv \det|A - \lambda I|$,
where det is the determinant function for matrices; it can be shown that the characteristic function depends only upon the operator A, and not on the specific matrix representation used for A. The solutions of the characteristic equation $c(\lambda) = 0$ are the eigenvalues of the operator A. By the fundamental theorem of algebra, every polynomial has at least one complex root, so every operator A has at least one eigenvalue, and a corresponding eigenvector. The eigenspace corresponding to an eigenvalue v is the set of vectors which have eigenvalue v. It is a vector subspace of the vector space on which A acts.
A diagonal representation for an operator A on a vector space V is a representation $A = \sum_i \lambda_i |i\rangle\langle i|$, where the vectors $|i\rangle$ form an orthonormal set of eigenvectors for A with corresponding eigenvalues λ_i. An operator is said to be diagonalizable if it has a diagonal representation. In the next section we will find a simple set of necessary and sufficient conditions for an operator on a Hilbert space to be diagonalizable. As an example of a diagonal representation, note that the Pauli Z matrix may be written

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

where the matrix representation is with respect to orthonormal vectors $|0\rangle$ and $|1\rangle$, respectively. Diagonal representations are sometimes also known as orthonormal decompositions.

6.6.1 Product

We will use alternative notation for the product between two vectors,

$$\langle a|b\rangle = \langle a, b\rangle \tag{6.31}$$

We will use,

$$\langle a|b\rangle = \langle b|a\rangle^* . \tag{6.32}$$

Distributive

$$\langle a|b + c\rangle = \langle a|b\rangle + \langle a|c\rangle \tag{6.33}$$

where $|a\rangle, |b\rangle, |c\rangle \in \mathcal{C}$.

6.6.2 Norm and inner product

Norm

$$\|\mathbf{a}\| \quad = \quad \sqrt{\sum_{k=0}^{n-1} (a_k)^* a_k} = \sqrt{\sum_{k=0}^{n-1} |a_k|^2} \tag{6.34}$$

and inner product

$$\|\mathbf{a}\|^2 \quad \equiv \quad \langle a|a\rangle = \sum_{k=0}^{n-1} (a_k)^* a_k = \sum_{k=0}^{n-1} |a_k|^2 \geq 0. \tag{6.35}$$

6.6.3 Definitions

used to decribe quantum states: let $a, b \in \mathbb{C}^2$ (go to Hilbert Space)

$$\text{ket} \rightarrow |a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{6.36}$$

$$\text{bra} \rightarrow \langle b| = |b\rangle^\dagger = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}^\dagger = \begin{pmatrix} b_1^* b_2^* \end{pmatrix} \tag{6.37}$$

$$\text{bra-ket} \rightarrow \langle a|b\rangle = \langle b|a\rangle^* \in \mathbb{C}^2 \tag{6.38}$$

$$\text{ket-bra} \rightarrow |b\rangle \langle a| = \begin{pmatrix} a_1 b_1^* & a_1 b_2^* \\ a_2 b_1^* & a_2 b_2^* \end{pmatrix} \tag{6.39}$$

6.7 Density operator

Density matrices can represent the states. Those matrices are called density operators and it is formed by the outer product of a state vector [9, pag. 28],

$$\rho = |\psi\rangle \langle\psi| \tag{6.40}$$

The $|\psi\rangle$ can be represented by a superposition (even of pure states). We have next properties hold for pure states,

Idempotent

$$\rho^2 = |\psi\rangle \langle\psi|\psi\rangle \langle\psi| = |\psi\rangle \langle\psi| = \rho \tag{6.41}$$

Trace=1

$$\mathrm{Tr}(\rho) = \sum_n \langle n|\rho|n\rangle = \sum_n \langle n|\psi\rangle \langle\psi|n\rangle = \sum_n \langle\psi|n\rangle \langle n|\psi\rangle = 1 \tag{6.42}$$

where $\{|n\rangle\}$ is a orthonormal basis.

Similarly

$$\mathrm{Tr}\big(\rho^2\big) = 1 \tag{6.43}$$

Hermiticity

$$\rho^\dagger = \big(\left|\psi\right\rangle \left\langle\psi\right| \big)^\dagger = \left|\psi\right\rangle \left\langle\psi\right| = \rho \tag{6.44}$$

Positive semidefinite

$$\left\langle\phi\right| \rho \left|\phi\right\rangle = \left\langle\phi|\psi\right\rangle \left\langle\psi|\phi\right\rangle = \left|\left\langle\phi|\psi\right\rangle\right|^2 \geq 0 \tag{6.45}$$

With the mixed states density matrices have other properties [9, pag. 28]. Now we will see ρ as a matrix:
We define a state

$$\left|\psi\right\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$$

now we consider eq. (6.40), then

$$\rho = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \begin{bmatrix} \alpha_0^* & \alpha_1^* & \dots & \alpha_N^* \end{bmatrix}$$

then

$$\rho = \begin{bmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* & \dots & \alpha_0\alpha_N^* \\ \alpha_1\alpha_0^* & |\alpha_1|^2 & \dots & \alpha_1\alpha_N^* \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_N\alpha_0^* & \alpha_N\alpha_1^* & \dots & |\alpha_N|^2 \end{bmatrix}$$

Example 10 *Consider next state*

$$\left|\psi_{AB}\right\rangle = \frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The density is,

$$\begin{aligned} \rho_{AB} &= \left|\psi_{AB}\right\rangle \left\langle\psi_{AB}\right| \\ \rho_{AB} &= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \right) \\ \rho_{AB} &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{6.46}$$

Usually, states are in pure state which means that we can precisely define their quantum state at every point in time. For example, if we initialize the single qubit $|q\rangle$ *in state* $|0\rangle$ *(which is common), and apply any gate* (H, X, Y, Z)*, we know our final state.*

Chapter 7

Applications and advance topics

The commutator between two operators A and B is defined to be

$$[A, B] \equiv AB - BA \tag{7.1}$$

If $[A, B] = 0$, that is, $AB = BA$, then we say A commutes with B. Similarly, the anti-commutator of two operators A and B is defined by

$$\{A, B\} \equiv AB + BA \tag{7.2}$$

we say A anti-commutes with B if $\{A, B\} = 0$. It turns out that many important properties of pairs of operators can be deduced from their commutator and anti-commutator.

Chapter 8

Exercises

1. Represent any complex number in the polar representation (go to Complex plane).

2. Compute AB and BA considering two matrices (go to Matrix multiplication),

$$A = \begin{pmatrix} 24 & 22 & 20 & 18 \\ 16 & 14 & 12 & 10 \\ 2 & 4 & 6 & 8 \end{pmatrix}, \tag{8.1}$$

$$B = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{pmatrix}, \tag{8.2}$$

3. Use the definition of dot product to show that each of the following two sets of vectors is orthogonal.

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \tag{8.3}$$

and

$$\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \tag{8.4}$$

4. Prove that

$$U(t_1, t_2) \equiv \exp\left[\frac{-iH(t_2 - t_1)}{\hbar}\right] \tag{8.5}$$

is unitary.

5. Prove that

$$|\psi(t_2)\rangle = e^{-i\frac{H}{\hbar}(t_2 - t_1)}|\psi(t_1)\rangle \tag{8.6}$$

is a solution of the time-independent Shrödinger equation, namely,

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle \tag{8.7}$$

where H is the Hamiltonian of the system, which represents the total energy of the system. We can consider it as independent time function, namely a constant.

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